§ Levi-Civita connection

定義 $\langle \nabla_X Y, Z \rangle =$

Metric compatible： $X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

and torsion free： $\nabla_X Y - \nabla_Y X = [X,Y]$ for all vector fields X，Y。

存在性與唯一性的證明對於理論的建立應該很重要，

在局部座標中(local chart) $\nabla_{\partial_i}\partial_j = \Gamma_{ij}^k \partial_k$ 其中 $\partial_i = \dfrac{\partial}{\partial x^i}$

且 $\nabla_{\partial_i} dx^j = -\Gamma_{ik}^j dx^k$ 也因此推出

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l})$$

§ Riemannian curvature tensor

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

在局部坐標系中 $R(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j})\dfrac{\partial}{\partial x^l} = R_{lij}^k \dfrac{\partial}{\partial x^k}$

令 $R_{klij} := g_{km} R_{lij}^m$ 則 $R_{klij} = \left\langle R(\dfrac{\partial}{\partial x^i}, \dfrac{\partial}{\partial x^j})\dfrac{\partial}{\partial x^l}, \dfrac{\partial}{\partial x^k} \right\rangle$

對任意向量場 X、Y、Z

$$R(X,Y)Z = -R(Y,X)Z, \quad i.e.\ R_{k\ell ij} = -R_{k\ell ji}, \tag{4.3.9}$$

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \quad i.e.\ R_{k\ell ij} + R_{kij\ell} + R_{kj\ell i} = 0, \tag{4.3.10}$$

$$\langle R(X,Y)Z, W\rangle = -\langle R(X,Y)W, Z\rangle, \quad i.e.\ R_{k\ell ij} = -R_{\ell kij}, \tag{4.3.11}$$

$$\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle, \quad i.e.\ R_{k\ell ij} = R_{ijk\ell}. \tag{4.3.12}$$

Bianchi 第二恆等式

$\nabla_{\partial_h} R_{klij} + \nabla_{\partial_k} R_{lhij} + \nabla_{\partial_l} R_{hkij} = 0$ 注意到 h k l 的循環，R 的最後兩個 index 都是 ij

§ 截曲率

黎曼流形上兩切向量 $X = \xi^i \dfrac{\partial}{\partial x^i}, Y = \eta^i \dfrac{\partial}{\partial x^i}$ 所張平面的截曲率

$$K(X \wedge Y) := \langle R(X,Y)Y, X\rangle \frac{1}{|X \wedge Y|^2}$$

Ricci tensor $R_{ij} = g^{kj} R_{ikjl}$ and scalar curvature $R = g^{ij} R_{ij}$

**Definition 4.3.4** The Riemannian manifold $M$ is called a space of constant sectional curvature, or a *space form* if $K(X \wedge Y) = K \equiv$ const. for all linearly independent $X, Y \in T_x M$ and all $x \in M$. A space form is called *spherical*, *flat*, or *hyperbolic*, depending on whether $K > 0, = 0, < 0$.
$M$ is called an *Einstein manifold* if

$$R_{ik} = c g_{ik}, \quad c \equiv \text{const.}$$

(note that $c$ does not depend on the choice of local coordinates).

Schur theorem
Let $d = \dim M \geq 3$
If the sectional curvature of M is constant at each point，i.e. $K(X \wedge Y) = f(x)$ for $X, Y \in T_x M$ then f(x)≡const and M is a space form。
A space form is called spherical，flat，or hyperbolic， depending on whether K >0，=0，< 0

Likewise，if the Ricci curvature is constant at each point，i.e. $R_{ij} = c(x) g_{ij}$ then $c(x) \equiv$ const and M is Einstein。
Schu theorem says that the isotropy(各向同性) of a Riemannian manifold，i.e. the property that at each point all directions are geometrically indistinguishable，implies the homogeneity， i.e. that all points are geometrically indistinguishable。In particular，a pointwise property implies a global one。