§ Hessian

(M,g) is a Riemannian manifold，$f : M \to R$ is a differentiable function

The Hessian of f is ∇df

$$df = \frac{\partial f}{\partial x^i} dx^i \quad \text{then} \quad \nabla df = (\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial x^k} \Gamma_{ij}^k) dx^i dx^j$$

Hessian 稱為黑塞矩陣 在 R^2，$H(x, y) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix}$

定義

1. $(Hessf)(X, Y) = g(\nabla_X(gradf), Y)$
2. $(Hessf)(X, Y) = (\nabla_X df)Y = X(Y(f)) - (\nabla_X Y)f$

$z = f(x, y) = x^2 + xy + y^2 \quad \begin{cases} \dfrac{\partial f}{\partial x} = 0 \\ \dfrac{\partial f}{\partial y} = 0 \end{cases} \Rightarrow (x, y) = (0,0)$

$H = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ detH>0 且 $f_{xx} = 2 > 0$ 所以 f(x,y)在(0,0)有極小值。