§ The Laplace Operator on Forms

Hodge star operator

Lemma 3.3.1

$**=(-1)^{p(n-p)}:\Lambda^p(V)\to\Lambda^p(V)$

Lemma 3.3.2 For $v,w\in\Lambda^p(V)$

$\langle v,w\rangle=*(w\wedge *v)=*(v\wedge *w)$

定義

Definition 3.3.1 d^* is the operator which is (formally) adjoint to d on $\overset{d}{\underset{p=0}{\oplus}}\Omega^p(M)$ w.r.t. $(\cdot,\cdot)$. This means that for $\alpha\in\Omega^{p-1}(M)$, $\beta\in\Omega^p(M)$

$$(d\alpha,\beta)=(\alpha,d^*\beta); \tag{3.3.11}$$

d^* therefore maps $\Omega^p(M)$ to $\Omega^{p-1}(M)$.

Lemma 3.3.4

$d^*:\Omega^p(M)\to\Omega^{p-1}(M)$ satisfies $d^*=(-1)^{n(p+1)+1}*d*$

定義

Definition 3.3.2 The *Laplace(–Beltrami) operator* on $\Omega^p(M)$ is

$$\Delta=dd^*+d^*d:\Omega^p(M)\to\Omega^p(M).$$

$\omega\in\Omega^p(M)$ is called *harmonic* if

$$\Delta\omega=0\,.$$

推論

Δ is(formally)selfadjoint i.e. $(\Delta\alpha,\beta)=(\alpha,\Delta\beta)$ for $\alpha,\beta\in\Omega^p(M)$

Lemma 3.3.5

$$(\Delta\alpha,\alpha)=(dd^*\alpha,\alpha)+(d^*d\alpha,\alpha)=(d^*\alpha,d^*\alpha)+(d\alpha,d\alpha)\geq 0. \tag{3.3.13}$$

In particular, Δ is nonnegative, and

$$\Delta\alpha=0 \textit{ iff } d\alpha=0 \textit{ and } d^*\alpha=0. \tag{3.3.14}$$

Corollary 3.3.2 *On a compact Riemannian manifold, every harmonic function is constant.* □

Lemma 3.3.6 $*\Delta=\Delta *$.

Theorem 3.3.1

$$\sigma(\Delta_p)\backslash\{0\} = \sigma(D'_{p-1})\backslash\{0\} \cup \sigma(D'_p)\backslash\{0\}. \qquad (3.3.26)$$

□

Corollary 3.3.3 *When we know the spectra of Δ_{p-1} and Δ_{p+1}, then we also know the one of Δ_p, except for the multiplicity of the eigenvalue 0, that is, the number of linearly independent harmonic forms.* □

§ 2.餘微分

1.1 定義

在黎曼幾何中，外微分算子 d 在 L^2 內積下的伴隨算子，正式名稱為**餘微分**(codifferential)，通常記為 δ 。

定義

在一個 n 維、緊緻、無邊界的黎曼流形(M,g)上，我們可以在微分形式空間 $\Omega^k(M)$ 上定義 L^2 內積：$\langle\omega,\eta\rangle = \int_M \omega\wedge *\eta$

這裡∗是 Hodge star operator，它是將 k 形式映射為 (n-k)-形式的同構。

δ 定義為 $\langle d\omega,\theta\rangle = \langle\omega,\delta\theta\rangle$

經過分部積分（斯托克斯定理）推導，可以得到 δ 的顯式公式

$$\delta = (-1)^{n(k+1)+1} * d *$$

1.2 R^3 中的例子

在 R^3 中，Hodge star operator：

0-form f $*f = f dx\wedge dy\wedge dz$

1-form $*dx = dy\wedge dz \quad *dy = dz\wedge dx \quad *dz = dx\wedge dy$

2-form $*(dy\wedge dz) = dx$

3-form $*(dx\wedge dy\wedge dz) = 1$

例 1

δ 作用於 1-form 相當於負的散度。 $\varphi = \sum_i \varphi_i dx^i$ 則 $d^*\varphi = -\sum_i \frac{\partial\varphi_i}{\partial x^i} = -div\varphi$

假設 $\omega = A_x dx + A_y dy + A_z dz$

先取 $*\omega$ $\quad *\omega = A_x(dy\wedge dz) + A_y(dz\wedge dx) + A_z(dx\wedge dy)$

外微分 $d*\omega$ $\quad d*\omega = \frac{\partial A_x}{\partial x} dx\wedge dy\wedge dz + \frac{\partial A_y}{\partial y} dy\wedge dz\wedge dx + \frac{\partial A_z}{\partial z} dz\wedge dx\wedge dy$

$$=(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z})dx\wedge dy\wedge dz$$

取$*d*\omega$ $\quad *d*\omega=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=\nabla\cdot A$

在R^3，對於 1-form k=1，$\delta\omega=(-1)^k*d*\omega=-\nabla\cdot A$

例 2

δ作用於 2-form 相當於旋度。

假設$\eta=B_x dy\wedge dz+B_y dz\wedge dx+B_z dx\wedge dy$

$$*\eta=B_x dx+B_y dy+B_z dz$$

$$d(*\eta)=(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z})dy\wedge dz+...$$

$$*d*\eta=(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z})dx+...$$

2-form時，k=2，$\delta\eta=*d*\eta=(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z})dx+...=(\nabla\times B)^b$

1.3 在二維球面S^2上的例子

$$g=d\theta^2+\sin^2\theta d\phi^2$$

0-form $*1=\sin\theta d\theta\wedge d\phi$ (volume form)

1-form $*d\theta=\sin\theta d\phi \qquad *d\phi=-\frac{1}{\sin\theta}d\theta$

2-form $*(d\theta\wedge d\phi)=\frac{1}{\sin\theta}$

例 $\alpha=A(\theta,\phi)d\theta+B(\theta,\phi)d\phi$

…

$$\delta\alpha=-*d*\alpha=-(\frac{\partial A}{\partial\theta}+\cot\theta A+\frac{1}{\sin^2\theta}\frac{\partial B}{\partial\phi})$$

§ 03 Hodge-Laplacian

3.1 定義

$\Box=d\delta+\delta d$ (Hodge-Laplacian or Laplace-de Rham operator)

滿足 $\Box\omega=0$的differential form稱為harmonic form。

3.2 例子

例 $f(x,y,z)=x^2+y^2+z^2$是一個0-form $\delta f=0$

$$\Box f=(d\delta+\delta d)f$$

$$df=2xdx+2ydy+2zdz$$

$$\delta(df)=-(\frac{\partial(2x)}{\partial x}+\frac{\partial(2y)}{\partial y}+\frac{\partial(2z)}{\partial z})=-6$$

所以$\Box f=-6$

例 $\alpha=(y+z)dx+(x+z)dy+(x+y)dz$

$$\delta=(-1)^{n(k+1)+1}*d*$$

$$\delta\alpha=-(\frac{\partial(y+z)}{\partial x}+\ldots)=0$$

$$d\alpha=d(y+z)\wedge dx+d(x+z)\wedge dy+d(x+y)\wedge dz=\ldots=0$$

$\Box\alpha=0$所以α是一個 harmonic form。

例 S^2上 $g=d\theta^2+\sin^2\theta d\phi^2$ $e^1=d\theta,e^2=\sin\theta d\phi$

0- form $*1=\sin\theta d\theta\wedge d\phi$

1- form $*d\theta=\sin\theta d\phi$ $*d\phi=-\frac{1}{\sin\theta}d\theta$

2- form $*(d\theta\wedge d\phi)=\frac{1}{\sin\theta}$

$\alpha=A(\theta,\phi)d\theta+B(\theta,\phi)d\phi$

…

$$\delta\alpha=-(\frac{\partial A}{\partial\theta}+\cot\theta A+\frac{1}{\sin^2\theta}\frac{\partial B}{\partial\phi})$$

[Ex] $\omega=\sin\theta d\phi,\sin\theta d\theta$ 分別求$\delta\omega$

3.3 一個恆等式

對任意$v,w\in\Lambda^p(V)$ 證明$\langle v,w\rangle=*(w\wedge *v)=*(v\wedge *w)$

附錄

變分法推導$\Delta f=0$的詳細過程

$f:\mathrm{M}\to\mathrm{R}$ is a smooth function

The enrggy functional is

$$\mathrm{E}(f):=\frac{1}{2}\int_M<df,df>dV=\frac{1}{2}\int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\sqrt{g}dx^1\ldots dx^n$$

Where $\sqrt{g} = \sqrt{\det(g_{ij})}$

If $\frac{d}{dt}E(f+t\eta)|_{t=0}=0$ for all $\eta : M \to R$ with compact support

Prove that $\Delta f = 0$

Consider the variation $f_t = f + t\eta$ then

$$\mathrm{E}(f_t) = \frac{1}{2}\int_M g^{ij}\frac{\partial(f+t\eta)}{\partial x^i}\frac{\partial(f+t\eta)}{\partial x^j}\sqrt{g}dx^1 \ldots dx^n = \frac{1}{2}\int_M g^{ij}(\frac{\partial f}{\partial x^i}+t\frac{\partial \eta}{\partial x^i})(\frac{\partial f}{\partial x^i}+t\frac{\partial \eta}{\partial x^i})\sqrt{g}dx^1 \ldots dx^n$$

$$\frac{d}{dt}E(f_t)|_{t=0} = \frac{1}{2}\int_M g^{ij}[\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}+\frac{\partial \eta}{\partial x^i}\frac{\partial f}{\partial x^j}]\sqrt{g}dx^1 \ldots dx^n = \int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}\sqrt{g}dx^1 \ldots dx^n$$

$$g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j} = <\nabla f, \nabla \eta>$$

$\frac{d}{dt}E(f_t)|_{t=0} = \int_M <\nabla f, \nabla \eta> dV = 0$ for all η

By integration by parts

On a Riemannian manifold, $div(uX) = udiv(X) + <\nabla u, X>$

Where X is a vector field and u is a function, then

$$\int_M div(uX)dV = \int_M udivXdV + \int_M <\nabla u, X> dV$$

$\int_M div(uX)dV = \int_{\partial M} <uX, n> dS = 0$ (by divergence theorem, $u|_{\partial M} = 0$)

Let $u = \eta, X = \nabla f$ then $\int_M <\nabla f, \nabla \eta> dV = -\int_M \eta div(\nabla f)dV = -\int_M \eta \Delta f dV$

$\int_M \eta \Delta f dV = 0$ for all function η with compact support, this implies

$\Delta f = 0$