

§ 3.2 The Spectrum of the Laplace Operator [GA3.2]

§ Rellich Embedding Theorem

找出一個區域 $\Omega \subset \mathbf{R}^n$ ，熱平衡時的溫度分布 $u(x)$ 滿足 $-\Delta u = f(x)$, in Ω
 $u=0$, on $\partial\Omega$ 其中 $f(x) \in L^2(\Omega)$ ，表示某種內部熱源。

這是一個經典的 Poisson 方程。

我們不直接解微分方程，而是從變分角度來求解。

考慮泛函 $J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} f u dx$ 的最小函數 u :

我們考慮的解空間是 Sobolev 空間： $H_0^1(\Omega) = \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega), u|_{\partial\Omega} = 0\}$

「如果 $\{u_k\}$ 在 $H_0^1(\Omega)$ 中有界，則它在 $L^2(\Omega)$ 中會有收斂子序列。」

這是 embedding 的內涵：

一個在 Sobolev 空間裡有界的函數列，可以在 L^2 空間中找出一個收斂子列。

我們現在有一列函數 $\{u_k\} \subset H_0^1(\Omega)$ ，是這個泛函的近似最小值序列，那麼這些近似解會收斂到某個真正的解。

例 $u_k(x) = \frac{\sqrt{2}}{k\pi} \sin(k\pi x)$

$$\|u_k\|_{H^1}^2 = \|u_k\|_{L^2}^2 + \|u_k'\|_{L^2}^2 = \int_0^1 |u_k|^2 dx + \int_0^1 |u_k'|^2 dx = \frac{1}{(k\pi)^2} + 1 \leq 2$$

那麼 $\{u_k(x)\}$ 在 L^2 空間中有一個收斂子序列。

The function space is the Sobolev $H := H^{1,2}(M)$

L^2 -product $(f, g) := \int_M f(x)g(x)\sqrt{g}dx^1 \dots dx^n$ for $f, g \in L_2(M)$

And $\|f\| := (f, f)^{\frac{1}{2}}$

$(\Delta\varphi, \psi) = (d\varphi, d\psi) = (\varphi, \Delta\psi)$

We define $\lambda_1 := \inf_{f \in H, \int_M f=0} \frac{(df, df)}{(f, f)}$, then $\lambda_1 > 0$ (from the Poincare inequality)

$\Delta f(x) = \lambda f(x)$ for all $x \in \Omega$

§ Theorem 3.2.1

Let M be a compact Riemannian manifold.

The eigenvalue problem $\Delta f = \lambda f, f \in H^{1,2}$ has countably many eigenvalues with

pairwise orthonormal vectors v_m °

$$(v_m, v_l) = \delta_{ml}, \Delta v_m = \lambda_m v_m, (dv_m, dv_l) = \lambda_m \delta_{ml}$$

Except for the eigenvalue $\lambda_0 = 0$ realized for a constant as its eigenfunction, all

eigenvalues are positive and $\lim_{m \rightarrow \infty} \lambda_m = \infty$

For $f \in L^2(M)$, we have $f = \sum_{i=0}^{\infty} (f, v_i) v_i$, where this series converges in L^2 and if

$$f \in H^{1,2}(M) \text{ we also have } (df, df) = \sum_{i=1}^{\infty} \lambda_i (f, v_i)^2$$

1. 離散的譜與無窮特徵值
對於一個緊緻 (Compact) 的黎曼流形 M , 其拉普拉斯算子的特徵值並非連續的分佈, 而是離散的(有無窮多個)。
2. 特徵函數的完備正交性
 $(v_m, v_n) = \delta_{mn}$
3. 第一特徵值與常數函數
 $\lambda_0 = 0$ (對應常數函數)

The eigenvalues and eigenfunctions of the Laplace–Beltrami operator of a Riemannian manifold M determine its Green function and heat kernel. We have

$$\Gamma(x, y) = \sum_{j \geq 1} \frac{1}{\lambda_j} v_j(x) v_j(y) \quad (3.2.21)$$

and

$$p(x, y, t) = \sum_{i > 1} e^{-\lambda_i t} v_i(x) v_i(y). \quad (3.2.22)$$

The eigenvalues of the Laplace-Beltrami operator of a Riemannian manifold M encodes its geometric quantities, and conversely, those eigenvalues can be estimated in terms of geometric quantities °

1. Weyl estimate
[Spec802]
2. Cheeger estimate

對於一個 compact 黎曼流形 M , $h(M) = \inf_A \frac{Area(\partial A)}{\min(vol(A), vol(B))}$

$$\lambda_1 \geq \frac{h(M)^2}{4}$$

Selberg 猜想：對於任何同餘子群 (congruence subgroup) Γ , 其 Laplace-Beltrami

算子的 $\lambda_1, \lambda_1 \geq \frac{1}{4}$

在雙曲幾何中...

3. Federer formula

Theorem For every open $\Omega \subset M$ and every smooth function ϕ on Ω , we have

$$\int_{\Omega} \|d\phi\| = \int_{-\infty}^{\infty} \text{Vol}_{d-1}(\phi^{-1}(t) \cap \Omega) dt, \quad (3.2.31)$$

where Vol_{d-1} indicates the $(d-1)$ -dimensional volume of a hypersurface. \square