§ 譜與 Sobolev 空間

Sobolev space $H := H^{1,2}(M)$

An eigenfunction f of Δ is a function $f \in H, f \neq 0$，that satisfies $\Delta f(x) = \lambda f(x)$ for all $x \in \Omega$

All eigenvalues λ are nonnegative。

1. Integration by parts (or Green Identity)

$$\int_M < \nabla u, \nabla v > dV + \int_M u \Delta v dV = \int_{\partial M} u \frac{\partial v}{\partial n} dS$$

2. $div(f\Delta f) = \nabla f \bullet \nabla f + f div(\Delta f)$

$\Delta = -div(grad) \Rightarrow div(\nabla f) = -\Delta f$

$div(f\Delta f) = \nabla f \bullet \nabla f + f div(\Delta f) = \nabla f \bullet \nabla f - f\Delta f$

$\int_M div(f\Delta f)dV = \int_{\partial M} (f\Delta f) \bullet n dS = 0$ (Stokes theorem and M is compact，no boundary)

$\therefore \int_M (|\nabla f|^2 - f\Delta f) dV = 0$

$\int_M f\Delta f dV = \lambda \int_M f^2 dV$

$\int_M f\Delta f dV = \int_M < \nabla f, \nabla f > dV = \int_M |\nabla f|^2 dV$，we have $\lambda \int_M f^2 dV = \int_M |\nabla f|^2 dV$

$\therefore \lambda \geq 0$

Let $\Omega \subseteq \mathrm{R}^n$ be an open set。$1 \leq p \leq \infty, k \in N$

The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable

functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \leq k$，

the weak derivative $D^\alpha u$ exists and belong to $L^p(\Omega)$。

The norm on $W^{k,p}(\Omega)$ is given by：

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, 1 \leq p < \infty$$

And for $p = \infty$：$\|u\|_{W^{k,p}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$

其中 $H^1(\Omega) = W^{1,2}(\Omega)$ 在 Laplace equation 中很重要。