§ Laplace Operator on Functions [GA3.1]

§ 梯度 散度與Laplace算子

1. 梯度 散度與對偶

設 $f:R^n \to R$ 為光滑函數，定義 梯度 $\nabla f = grad(f) = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}$ ，

外微分得 1-form $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$ 。

兩者通過對偶關係聯繫：對任意向量場 X，$df(X) = <\nabla f, X>$ 。

散度：For a vector field $Z = \sum_i Z^i \frac{\partial}{\partial x^i}$ ，$div(Z) = \sum_i \frac{\partial Z^i}{\partial x^i}$ 。

For a 1-form $\varphi = \sum_i \varphi_i dx^i$ ，定義餘微分算子為 $d^*\varphi := -\sum_i \frac{\partial \varphi^i}{\partial x^i} = -div\varphi$

$\varphi^i = g^{ij}\varphi_j$ 在 R^n 中上下標在數值上是相同的。

d^*是外微分 d 的 L^2-adjoint $\Leftrightarrow \left\langle d\alpha, \beta \right\rangle_{L^2} = \left\langle \alpha, d^*\beta \right\rangle_{L^2}$ ，$d^* = (-1)^{n(p+1)+1} * d *$

例 R^3 中 $\varphi = (x^2 z)dx + (y^3 + \sin x)dy + (e^z \cos x)dz$

$d^*\varphi := -\sum_i \frac{\partial \varphi^i}{\partial x^i} = -div\varphi = -(2xz + 3y^2 + e^z \cos x)$

散度定理：§ For a compact supported 1-form φ，we have

$\int div\varphi dx^1 \wedge ... \wedge dx^n = 0$ (Stokes theorem)

一般而言 $(df, \varphi) = \int_{R^n} \sum_i \frac{\partial f}{\partial x^i}\varphi_i dx^1 \wedge ... \wedge dx^n = -\int_{R^n} f \frac{\partial \varphi^i}{\partial x^i} dx^1 \wedge ... \wedge dx^n$ 。

For a smooth compactly supported function f，we have $\int \Delta f dx^1 ... dx^n = 0$ ，

$(\Delta f, g) = (df, dg) = (f, dg)$

註：

$(\Delta f, g) = \int (\Delta f) g dx = \int (-\sum_i \frac{\partial^2 f}{\partial x_i^2}) g dx = \int \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x^i} dx = (df, dg)$

在歐氏空間中，這等價於 $\int_\Omega \nabla \cdot F dV = \int_{\partial\Omega} F \cdot n dS$ ，若 Ω 無邊界或 F 在邊界為零，則右式=0

例 在圓盤 $D = \left\{ (x,y) \middle| x^2 + y^2 \le 1 \right\}$ 上取 $\varphi = (1 - x^2 - y^2)(xdx + ydy)$

對應於向量場 $F = ((1-r^2)x, (1-r^2)y), r^2 = x^2 + y^2$ 。

計算散度 $\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} = ... = 2 - 4r^2$

計算積分 $\int_D \nabla \cdot F dA = \int_0^{2\pi} \int_0^1 (2-4r^2) r dr d\theta = 0$

2. Laplace-Beltrami 算子

在黎曼流形上，Laplace-Beltrami operator 定義為

$\Delta f := -divgradf = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j})$ 。其中 $\sqrt{g} = \sqrt{\det(g_{ij})}$

此定義與歐氏空間中的拉普拉斯算子相差一個負號，目的是使算子非負（特徵值非負）。

3. Dirichlet 能量與調和函數

定義函數 f:$\mathrm{M} \to \mathrm{R}$ 的能量泛函(Dirichlet integral)：

$\mathrm{E}(f) := \frac{1}{2}\int_M < df, df > dV = \frac{1}{2}\int_M g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\sqrt{g} dx^1 ... dx^n$ ，其中

$\sqrt{g} = \sqrt{\det(g_{ij})}, dV = \sqrt{g} dx^1 ... dx^n$

Lemma 3.1.1

若 f 是能量泛函的臨界點，即對所有緊支撐變分 $\eta : M \to R$

$\frac{d}{dt} E(f + t\eta)|_{t=0} = 0$ 則 f 滿足 $\Delta f = 0$ 稱為調和函數。

證明 令 $f_t = f + t\eta$ 則

$$\mathrm{E}(f_t) = \frac{1}{2}\int_M g^{ij} \frac{\partial (f+t\eta)}{\partial x^i}\frac{\partial (f+t\eta)}{\partial x^j}\sqrt{g} dx^1 ... dx^n = \frac{1}{2}\int_M g^{ij} (\frac{\partial f}{\partial x^i} + t\frac{\partial \eta}{\partial x^i})(\frac{\partial f}{\partial x^i} + t\frac{\partial \eta}{\partial x^i})\sqrt{g} dx^1 ... dx^n$$

$$\frac{d}{dt} E(f_t)|_{t=0} = \frac{1}{2}\int_M g^{ij} [\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j} + \frac{\partial \eta}{\partial x^i}\frac{\partial f}{\partial x^j}]\sqrt{g} dx^1 ... dx^n = \int_M g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}\sqrt{g} dx^1 ... dx^n$$

$$g^{ij} \frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j} = < \nabla f, \nabla \eta >$$

$\frac{d}{dt} E(f_t)|_{t=0} = \int_M < \nabla f, \nabla \eta > dV = 0$ for all η

By integration by parts

On a Riemannian manifold，$div(uX) = udiv(X) + < \nabla u, X >$

Where X is a vector field and u is a function，then

$$\int_M div(uX) dV = \int_M udivX dV + \int_M < \nabla u, X > dV$$

$\int_M div(uX) dV = \int_{\partial M} < uX, n > dS = 0$ (by divergence theorem，$u|_{\partial M} = 0$)

Let $u=\eta, X=\nabla f$ then $\int_M <\nabla f,\nabla\eta> dV = -\int_M \eta div(\nabla f)dV = -\int_M \eta\Delta f dV$

$\int_M \eta\Delta f dV = 0$ for all function η with compact support，this implies

$\Delta f = 0$

4. Laplacian 的幾何意義

§ 一維的情形

考慮原點的鄰域 (-h,h)，$u: R\to R$

$$u(x)=u(0)+u'(0)x+\frac{x^2}{2}u''(0)+\frac{x^3}{6}u'''(0)+o(x^4)$$

$$\bar{u}(h)=\frac{1}{2h}\int_{-h}^{h}u(x)dx=u(0)+\frac{u''(0)}{6}h^2+o(h^4)$$

Then $u''(0)=\frac{6}{h^2}(\bar{u}(h)-u(0))+o(h^2)$

u 在 O 點附近的平均值與 u(0)的差量就是 u(x)的 Laplacian 的幾何意義。

$$\Delta u(x)=\frac{d^2u}{dx^2}$$

§ 二維的情形

考慮 $D_h=\{(x,y)\,|\,x^2+y^2\le h^2\}$

平均值的定義有兩種：(1) $\frac{1}{\pi r^2}\iint_{B_r}u(x,y)dxdy$ (2) $\frac{1}{2\pi r}\oint_{\partial B_r}u(x,y)dxdy$

u(x,y)在(0,0)展開

$$u(x,y)=u(0,0)+\frac{\partial u}{\partial x}(0,0)x+\frac{\partial u}{\partial y}(0,0)y+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(0,0)x^2+\frac{\partial^2 u}{\partial x\partial y}(0,0)xy+\frac{1}{2}\frac{\partial^2 u}{\partial y^2}(0,0)y^2+o(r^2)$$

其中 $\iint_{D_h}xdA=\iint_{D_h}ydA=0$ ， $\iint_{D_h}xydA=0$

$\iint_{D_h}x^2dA=\iint_{D_h}y^2dA$ 合併計算 $\iint_{D_h}(x^2+y^2)dA=\int_0^{2\pi}\int_0^h r^2\cdot rdrd\theta=\ldots=\frac{\pi h^4}{2}$

,,,

$$\bar{u}(h)=\frac{1}{\pi h^2}\iint_{D_k}u(x,y)dA=u(0,0)+\frac{h^2}{8}(u_{xx}(0,0)+u_{yy}(0,0))+o(h^2)$$

$\Delta u(0,0)=\frac{8}{h^2}(\bar{u}(h)-u(0,0))+o(1)$ $o(1)\to 0$ as $h\to 0$

$$\Delta u(x,y)=u_{xx}+u_{yy}$$

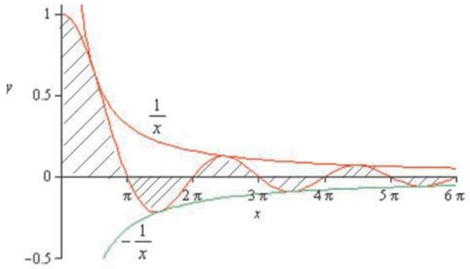
所以 Laplacian 量化了函數在該點附近平均值與中心點的偏差。

5. Dirichlet Integral 兩種不同的概念：

   (1) 分析中的廣義函數(sinc 函數)

   (2) 變分法與 PDE 中的能量泛函

定義 1: 經典反常(improper)積分 $\int_0^\infty \frac{\sin x}{x} = \qquad \frac{\pi}{2}$



左圖 $y = \frac{six}{x}$ (也稱為 sinc 函數)

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[magazine/EvenSin]

可推出 $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

§ Feymann 的密技

考慮 $I(a) = \int_0^\infty e^{-ax} \times \frac{\sin x}{x} dx$

$$\frac{dI}{da} = \int_0^\infty -xe^{-ax} \times \frac{\sin x}{x} dx = -\int_0^\infty e^{-ax} \sin x dx = \ldots = \frac{-1}{1+a^2}$$

$I(a) = C - \arctan a \quad I(a) \to 0$ as $a \to \infty$ 所以 $C = \frac{\pi}{2}$

最後 $a \to 0$ 得 $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

跟做 Fourier transform 一樣。

習作 $\int_0^\infty \frac{x - \sin x}{x^3} = \qquad \frac{\pi}{4}$

定義 2：能量泛函 $\mathrm{E}[u] = \frac{1}{2}\int_\Omega |\nabla u|^2 dV$

應用

1. Harmonic map 能量
2. 孤立子與非線性薛丁格方程(NLS)
3. Minimal surface 的能量泛函
4. 幾何流與熱方程
5. 算子譜與 Dirac 算子

在流形 M 上，$E[u] = \frac{1}{2}\int_M |\nabla u|^2 dV_g$ 這裡的梯度與體積元素都由度量 g 決定。

例 Dirichlet 原理說：在固定邊界值的所有函數中，調和函數最小化能量。

$\Omega = B_1(0) \subset R^2$ 單位圓盤

$u(x,y) = x^2 - y^2$ ，$\Delta u = 0$ 所以 u 是 harmonic 函數

$\int_\Omega |\nabla u|^2 \, dxdy = ... = 2\pi$

大哉問：

§ 如何在 Lean4 中形式化描述 harmonic map 與其 Euler-Lagrange 方程？