§ The Laplace Operator and Harmonic Differential Forms

幾何分析的一個主要課題是某些作用量(action)或能量泛函的臨界點。例如測地線。

§ 3.1 The Laplace operator on functions

考慮黎曼流形上的調和函數(harmonic functions)，更一般地，考慮調和函數據以定義的 Dirichlet integral，算子，或 Laplace-Beltremi 算子。

$f: R^n \to R$ 梯度 $\nabla f = grad(f) = \sum_i \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}$ ，1-form $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$

兩者是對偶關係：

$df(X) = < gradf, X >$ for every vector field X。

For a vector $Z = \sum_i Z^i \frac{\partial}{\partial x^i}$ ， $div(Z) = \sum_i \frac{\partial Z^i}{\partial x^i}$

For a 1-form $\varphi = \sum_i \varphi_i dx^i$ ， $d^*\varphi := -\sum_i \frac{\partial \varphi^i}{\partial x^i} = -div\varphi$

($\varphi: M \to N$ is a smooth map then push a vector field and pull back a 1-form $\varphi_* X =$ $\varphi * \omega$ 不要搞混了)。

For a compact supported 1-form φ ，we have $\int div\varphi dx^1 \wedge ... \wedge dx^n = 0$ (Stokes theorem)

In Riemannian geometry

(1) For a vctor field X， $div(X) = \nabla_\mu X^\mu$

(2) For a one-form ω ， $div\omega = \nabla^\mu \omega_\mu = g^{\mu\nu}\nabla_\mu \omega_\nu$

Hodge star operator in R^3

$* f = f dx \wedge dy \wedge dz$

$*dx = dy \wedge dz$ $*dy = dz \wedge dx$ $*dz = dx \wedge dy$

$*(dy \wedge dz) = dx$ $*(dz \wedge dx) = dy$ $*(dx \wedge dy) = dz$

$*(dx \wedge dy \wedge dz) = 1$

Then $div(\omega) = *d*\omega$

A compactly supported one-form ω ：在某一 compact support 之外皆為零的 1-form

例如 在 R^2 上， $\omega = f(x,y)dx + g(x,y)dy$ 其中 f 和 g 僅在單位圓盤 $x^2 + y^2 \le 1$ 內非零，在圓盤外全為零。此時 ω 的支撐集為單位圓盤，是 compact set。

§ Laplace operator Δ

就一黎曼度量，$\Delta f = -\sum_{i=1}^{n}\frac{\partial^2 f}{\partial x^2} = -div(gradf)$ (注意到這裡與歐氏空間差一個負號。)

The Laplace-Beltrami operator $\Delta f := -divgradf = -\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^j})$

定義函數 f:M $\rightarrow$ R 的能量：

$$E(f) := \frac{1}{2}\int_M < df, df > dV = \frac{1}{2}\int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\sqrt{g}dx^1...dx^n$$

Where $\sqrt{g} = \sqrt{\det(g_{ij})}$

若 $\frac{d}{dt}E(f+t\eta)\big|_{t=0} = 0$ 對所有的 $\eta : M \rightarrow R$ 成立則稱 f 是 E(f)的臨界點。

我們可能假設 M 是 compact Riemannian manifold 以避免在積分邊界問題。可是這樣會有一些 trivial 情形，所以我們假設上式對所有的有 compact surport 的 η 成立。

以變分法而言，上面的推導就是 Dirichlet 積分 E 的 Euler-Lagrange 方程。

Lemma：

A smooth critical point f of the energy integral E in the sense that

$\frac{d}{dt}E(f+t\eta)\big|_{t=0} = 0$ for all $\eta : M \rightarrow R$ with compact support in M is harmonic，i.e.

$\Delta f = 0$

(這裡有一個計算放在附錄)

§ 3.2 The spectrum of the Laplace operator

What a a Sobolev space？

Let $\Omega \subseteq R^n$ be an open set。$1 \le p \le \infty, k \in N$

The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \le k$，the weak derivative $D^\alpha u$ exists and belong to $L^p(\Omega)$。

The norm on $W^{k,p}(\Omega)$ is given by：

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\le k}\|D^\alpha u\|_{L^p(\Omega)}^p\right)^{1/p}, 1 \le p < \infty$$

And for $p=\infty$: $\|u\|_{W^{k,p}(\Omega)}=\max_{|\alpha|\le k}\|D^{\alpha}u\|_{L^{\infty}(\Omega)}$

其中 $H^1(\Omega)=W^{1,2}(\Omega)$ 在 Laplace equation 中很重要。
Sobolev space $H:=H^{1,2}(M)$
An eigenfunction f of Δ is a function $f\in H, f\neq 0$ ，that satisfies
$\Delta f(x)=\lambda f(x)$ for all $x\in\Omega$
All eigenvalues λ are nonnegative。

1. Integration by parts (or Green Identity) $\int_M <\nabla u,\nabla v> dV+\int_M u\Delta v dV=\int_{\partial M} u\frac{\partial v}{\partial n}dS$
2. $div(f\Delta f)=\nabla f\bullet\nabla f+f div(\Delta f)$

$\Delta=-div(grad)\Rightarrow div(\nabla f)=-\Delta f$

$div(f\Delta f)=\nabla f\bullet\nabla f+f div(\Delta f)=\nabla f\bullet\nabla f-f\Delta f$

$\int_M div(f\Delta f)dV=\int_{\partial M}(f\Delta f)\bullet n dS=0$ (Stokes theorem and M is compact，no boundary)

$\therefore\int_M(|\nabla f|^2-f\Delta f)dV=0$

$\int_M f\Delta f dV=\lambda\int_M f^2 dV$

$\int_M f\Delta f dV=\int_M <\nabla f,\nabla f> dV=\int_M|\nabla f|^2 dV$ ，we have $\lambda\int_M f^2 dV=\int_M|\nabla f|^2 dV$

$\therefore\lambda\ge 0$

名詞解釋：

1. L^2 function： $\|f\|_2=(\int_X|f(x)|^2)^{1/2}d\mu(x)<\infty$

   X is a measure space (often R^n with Lebesque measure)

   $\|f\|_2$ is the L^2-norm of f

   L^2 functions form a Hilbert space with the inner product

   $<f,g>=\int_X f(x)\overline{g(x)}d\mu(x)$

   $f(x)=\frac{1}{1+x^2}$ then $f\in L^2(R)$ and $g(x)=\frac{1}{x}$ on (0,1) $g\notin L^2(0,1)$

   (1) $f(x)=x^{-1/3}$ wth a weak singularity but still L^2

   (2) $f(x)=\frac{\sin x}{x}$ on $(1,\infty)$ $\in L^2(1,\infty)$

   (3) $f(x)=\begin{cases}x^{-1/4}, 0<x<1\\ e^{-x}, x\ge 1\end{cases}$ $\in L^2(0,\infty)$

(4) $f(x)=e^{1/x}$ on (0,1) $\notin L^2(0,1)$

2. Sobolev space

3. Energy functional

$E(u)=\frac{1}{2}\int_M |\nabla u|^2 dV_g$ is the Dirichlet energy $|\nabla u|_g=\sqrt{g(\nabla u,\nabla u)}$

$E(f)=\frac{1}{2}\int_M <df,df>\sqrt{g}dx^1...dx^n$ ， $f:M\to R$ a smooth function

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§ 附錄

f:M $\to$ R is a smooth function

The enrggy functional is $\mathrm{E}(f):=\frac{1}{2}\int_M <df,df>dV=\frac{1}{2}\int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\sqrt{g}dx^1...dx^n$

Where $\sqrt{g}=\sqrt{\det(g_{ij})}$

If $\frac{d}{dt}E(f+t\eta)|_{t=0}=0$ for all $\eta:M\to R$ with compact support

Prove that $\Delta f=0$

Consider the variation $f_t=f+t\eta$ then

$$\mathrm{E}(f_t)=\frac{1}{2}\int_M g^{ij}\frac{\partial(f+t\eta)}{\partial x^i}\frac{\partial(f+t\eta)}{\partial x^j}\sqrt{g}dx^1...dx^n=\frac{1}{2}\int_M g^{ij}(\frac{\partial f}{\partial x^i}+t\frac{\partial \eta}{\partial x^i})(\frac{\partial f}{\partial x^i}+t\frac{\partial \eta}{\partial x^i})\sqrt{g}dx^1...dx^n$$

$$\frac{d}{dt}E(f_t)|_{t=0}=\frac{1}{2}\int_M g^{ij}[\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}+\frac{\partial \eta}{\partial x^i}\frac{\partial f}{\partial x^j}]\sqrt{g}dx^1...dx^n=\int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}\sqrt{g}dx^1...dx^n$$

$$g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}=<\nabla f,\nabla\eta>$$

$\frac{d}{dt}E(f_t)|_{t=0}=\int_M <\nabla f,\nabla\eta>dV$ =0 for all η

By integration by parts

On a Riemannian manifold，$div(uX)=udiv(X)+<\nabla u,X>$

Where X is a vector field and u is a function，then

$$\int_M div(uX)dV=\int_M udivXdV+\int_M <\nabla u,X>dV$$

$\int_M div(uX)dV=\int_{\partial M}<uX,n>dS=0$ (by divergence theorem，$u|_{\partial M}=0$)

Let $u=\eta, X=\nabla f$ then $\int_M <\nabla f,\nabla\eta>dV=-\int_M \eta div(\nabla f)dV=-\int_M \eta\Delta f dV$

$\int_M \eta\Delta f dV=0$ for all function η with compact support，this implies $\Delta f=0$