§ Laplace Operator [GA3.1]

§ 01 Laplace 算子作用於函數

1.1 梯度散度與對偶

設 $f:R^n\to R$ 為光滑函數，定義梯度 $\nabla f=grad(f)=\sum_i\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}$ ，外微分得 1-form $df=\sum_i\frac{\partial f}{\partial x^i}dx^i$ 。

兩者通過對偶關係聯繫：

對任意向量場 X， $df(X)=<gradf,X>$ 。

For a vector field $Z=\sum_i Z^i\frac{\partial}{\partial x^i}$ ， $div(Z)=\sum_i\frac{\partial Z^i}{\partial x^i}$ 。

For a 1-form $\varphi=\sum_i\varphi_i dx^i$ ，定義餘微分算子(伴隨)為

$$d^*\varphi:=-\sum_i\frac{\partial\varphi^i}{\partial x^i}=-div\varphi$$

1.2 散度定理

For a compact supported 1-form φ ，we have $\int div\varphi dx^1\wedge\ldots\wedge dx^n=0$

(Stokes theorem)

在歐氏空間相當於 $\int_\Omega\nabla\cdot FdV=\int_{\partial\Omega}F\cdot ndS$ ，若 Ω 無邊界或 F 在邊界為零，則右式=0

例　在圓盤 $D=\left\{(x,y)\middle|x^2+y^2\le 1\right\}$ 上取 $\varphi=(1-x^2-y^2)(xdx+ydy)$

對應於向量場 $F=((1-r^2)x,(1-r^2)y),r^2=x^2+y^2$ 。計算散度

$$\nabla\cdot F=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}=\ldots=2-4r^2$$

計算積分 $\int_D\nabla\cdot FdA=\int_0^{2\pi}\int_0^1(2-4r^2)rdrd\theta=0$

1.3 Laplace-Beltrami 算子

在黎曼流形上，The Laplace-Beltrami operator 定義為

$\Delta f:=-divgradf=-\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}(\sqrt{g}g^{ij}\frac{\partial f}{\partial x^j})$ 。其中 $\sqrt{g}=\sqrt{\det(g_{ij})}$

此定義與歐氏空間中的拉普拉斯算子相差一個負號，目的是使算子非負（特徵值非負）。

For a smooth compactly supported function f， $\int\Delta f dx^1\ldots dx^n=0$

1.4 Dirichlet 能量與調和函數

定義函數 f:M→R 的能量(稱為 Dirichlet integral)：

$$\mathrm{E}(f) := \frac{1}{2}\int_M < df, df > dV = \frac{1}{2}\int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\sqrt{g}dx^1...dx^n$$

Where $\sqrt{g} = \sqrt{\det(g_{ij})}$

若 f 是 E 的臨界點則對任意具緊支集的函數 $\eta : M \to R$，$\frac{d}{dt}E(f+t\eta)|_{t=0} = 0$

對所有 η 的成立。稱 f 為調和函數，且 $\Delta f = 0$

f:M→R is a smooth function

The enrggy functional is

$$\mathrm{E}(f) := \frac{1}{2}\int_M < df, df > dV = \frac{1}{2}\int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j}\sqrt{g}dx^1...dx^n$$

Where $\sqrt{g} = \sqrt{\det(g_{ij})}$

If $\frac{d}{dt}E(f+t\eta)|_{t=0} = 0$ for all $\eta : M \to R$ with compact support

Prove that $\Delta f = 0$

Consider the variation $f_t = f + t\eta$ then

$$\mathrm{E}(f_t) = \frac{1}{2}\int_M g^{ij}\frac{\partial(f+t\eta)}{\partial x^i}\frac{\partial(f+t\eta)}{\partial x^j}\sqrt{g}dx^1...dx^n = \frac{1}{2}\int_M g^{ij}(\frac{\partial f}{\partial x^i}+t\frac{\partial \eta}{\partial x^i})(\frac{\partial f}{\partial x^i}+t\frac{\partial \eta}{\partial x^i})\sqrt{g}dx^1...dx^n$$

$$\frac{d}{dt}E(f_t)|_{t=0} = \frac{1}{2}\int_M g^{ij}[\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}+\frac{\partial \eta}{\partial x^i}\frac{\partial f}{\partial x^j}]\sqrt{g}dx^1...dx^n = \int_M g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j}\sqrt{g}dx^1...dx^n$$

$$g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \eta}{\partial x^j} = < \nabla f, \nabla \eta >$$

$\frac{d}{dt}E(f_t)|_{t=0} = \int_M < \nabla f, \nabla \eta > dV = 0$ for all η

By integration by parts

On a Riemannian manifold，$div(uX) = udiv(X) + < \nabla u, X >$

Where X is a vector field and u is a function，then

$$\int_M div(uX)dV = \int_M udivXdV + \int_M < \nabla u, X > dV$$

$\int_M div(uX)dV = \int_{\partial M} < uX, n > dS = 0$ (by divergence theorem，$u|_{\partial M} = 0$)

Let $u = \eta, X = \nabla f$ then $\int_M < \nabla f, \nabla \eta > dV = -\int_M \eta div(\nabla f)dV = -\int_M \eta\Delta f dV$

$\int_M \eta\Delta f dV = 0$ for all function η with compact support，this implies

$\Delta f = 0$

1.5 譜與 Sobolev 空間

Sobolev space $H := H^{1,2}(M)$

An eigenfunction f of Δ is a function $f \in H, f \neq 0$，that satisfies
$\Delta f(x) = \lambda f(x)$ for all $x \in \Omega$

All eigenvalues λ are nonnegative。

1. Integration by parts (or Green Identity)

$$\int_M < \nabla u, \nabla v > dV + \int_M u \Delta v dV = \int_{\partial M} u \frac{\partial v}{\partial n} dS$$

2. $div(f\Delta f) = \nabla f \bullet \nabla f + f div(\Delta f)$

$\Delta = -div(grad) \Rightarrow div(\nabla f) = -\Delta f$

$div(f\Delta f) = \nabla f \bullet \nabla f + f div(\Delta f) = \nabla f \bullet \nabla f - f\Delta f$

$\int_M div(f\Delta f)dV = \int_{\partial M} (f\Delta f) \bullet n dS = 0$ (Stokes theorem and M is compact，no boundary)

$\therefore \int_M (|\nabla f|^2 - f\Delta f)dV = 0$

$\int_M f\Delta f dV = \lambda \int_M f^2 dV$

$\int_M f\Delta f dV = \int_M < \nabla f, \nabla f > dV = \int_M |\nabla f|^2 dV$，we have $\lambda \int_M f^2 dV = \int_M |\nabla f|^2 dV$

$\therefore \lambda \geq 0$

Let $\Omega \subseteq \mathrm{R}^n$ be an open set。$1 \leq p \leq \infty, k \in N$

The Sobolev space $W^{k,p}(\Omega)$ consists of all locally integrable

functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \leq k$，

the weak derivative $D^\alpha u$ exists and belong to $L^p(\Omega)$。

The norm on $W^{k,p}(\Omega)$ is given by：

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, 1 \leq p < \infty$$

And for $p = \infty$：$\|u\|_{W^{k,p}(\Omega)} = \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)}$

其中 $H^1(\Omega)=W^{1,2}(\Omega)$ 在 Laplace equation 中很重要。

§ 2.餘微分

2.1 定義

在黎曼幾何中，外微分算子 d 在 L^2 內積下的伴隨算子，正式名稱為**餘微分** (codifferential)，通常記為 δ 。

定義

在一個 n 維、緊緻、無邊界的黎曼流形(M,g)上，我們可以在微分形式空間 $\Omega^k(M)$ 上定義 L^2 內積：$\langle \omega,\eta\rangle=\int_M \omega\wedge *\eta$

這裡∗是 Hodge star operator，它是將 k 形式映射為 (n-k)-形式的同構。

δ 定義為 $\langle d\omega,\theta\rangle=\langle \omega,\delta\theta\rangle$

經過分部積分（斯托克斯定理）推導，可以得到 δ 的顯式公式

$\delta=(-1)^{n(k+1)+1} * d *$

2.2 R^3 中的例子

在 R^3 中，Hodge star operator：

0-form f $*f = f dx\wedge dy\wedge dz$

1-form $*dx=dy\wedge dz \quad *dy=dz\wedge dx \quad *dz=dx\wedge dy$

2-form $*(dy\wedge dz)=dx$

3-form $*(dx\wedge dy\wedge dz)=1$

例 1

δ 作用於 1-form 相當於負的散度。 $\varphi=\sum_i \varphi_i dx^i$ 則 $d^*\varphi=-\sum_i \frac{\partial\varphi_i}{\partial x^i}=-div\varphi$

假設 $\omega=A_x dx+A_y dy+A_z dz$

先取 $*\omega$ $\quad *\omega=A_x(dy\wedge dz)+A_y(dz\wedge dx)+A_z(dx\wedge dy)$

外微分 $d*\omega$

$$d*\omega=\frac{\partial A_x}{\partial x}dx\wedge dy\wedge dz+\frac{\partial A_y}{\partial y}dy\wedge dz\wedge dx+\frac{\partial A_z}{\partial z}dz\wedge dx\wedge dy$$

$$=(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z})dx\wedge dy\wedge dz$$

取 $*d*\omega$

$$*d*\omega=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}=\nabla\cdot A$$

在 R^3 ，對於 1-form k=1， $\delta\omega=(-1)^k * d*\omega=-\nabla\cdot A$

例 2

δ 作用於 2-form 相當於旋度。

假設 $\eta = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$

$*\eta = B_x dx + B_y dy + B_z dz$

$$d(*\eta) = (\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}) dy \wedge dz + ...$$

$$*d*\eta = (\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}) dx + ...$$

2-form 時，k=2， $\delta\eta = *d*\eta = (\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}) dx + ... = (\nabla \times B)^b$

2.3 在二維球面 S^2 上的例子

$g = d\theta^2 + \sin^2\theta d\phi^2$

0-form $*1 = \sin\theta d\theta \wedge d\phi$ (volume form)

1-form $*d\theta = \sin\theta d\phi$

$*d\phi = -\frac{1}{\sin\theta} d\theta$

2-form $*(d\theta \wedge d\phi) = \frac{1}{\sin\theta}$

例 $\alpha = A(\theta,\phi)d\theta + B(\theta,\phi)d\phi$

…

$$\delta\alpha = -*d*\alpha = -(\frac{\partial A}{\partial\theta} + \cot\theta A + \frac{1}{\sin^2\theta}\frac{\partial B}{\partial\phi})$$

§ 03 Hodge-Laplacian

3.1 定義

$\Box = d\delta + \delta d$ (Hodge-Laplacian or Laplace-de Rham operator)

滿足 $\Box\omega = 0$ 的 differential form 稱為 harmonic form。

3.2 例子

例 $f(x,y,z) = x^2 + y^2 + z^2$ 是一個 0-form $\delta f = 0$

$\Box f = (d\delta + \delta d) f$

$df = 2xdx + 2ydy + 2zdz$

$$\delta(df) = -(\frac{\partial(2x)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(2z)}{\partial z}) = -6$$

所以 $\Box f = -6$

例 $\alpha = (y+z)dx + (x+z)dy + (x+y)dz$

$\delta = (-1)^{n(k+1)+1} * d *$ [GA3.1-3]

$\delta\alpha = -(\frac{\partial(y+z)}{\partial x} + ...) = 0$

$d\alpha = d(y+z)\wedge dx + d(x+z)\wedge dy + d(x+y)\wedge dz = ... = 0$

$\Box\alpha = 0$所以α是一個 harmonic form。

例 S^2上 $g = d\theta^2 + \sin^2\theta d\phi^2$ $e^1 = d\theta, e^2 = \sin\theta d\phi$

0- form $*1 = \sin\theta d\theta \wedge d\phi$

1- form $*d\theta = \sin\theta d\phi$ $*d\phi = -\frac{1}{\sin\theta} d\theta$

2- form $*(d\theta\wedge d\phi) = \frac{1}{\sin\theta}$

$\alpha = A(\theta,\phi)d\theta + B(\theta,\phi)d\phi$

…

$\delta\alpha = -(\frac{\partial A}{\partial\theta} + \cot\theta A + \frac{1}{\sin^2\theta}\frac{\partial B}{\partial\phi})$

[Ex] $\omega = \sin\theta d\phi, \sin\theta d\theta$ 分別求$\delta\omega$

3.3 一個恆等式

對任意$v, w \in \Lambda^p(V)$ 證明$\langle v, w\rangle = *(w\wedge *v) = *(v\wedge *w)$

附錄

變分法推導$\Delta f = 0$的詳細過程

$f: \mathrm{M} \to \mathrm{R}$ is a smooth function

The enrggy functional is

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Where $\sqrt{g} = \sqrt{\det(g_{ij})}$

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$\frac{d}{dt}E(f_t)\big|_{t=0} = \int_M <\nabla f, \nabla \eta> dV = 0$ for all η

By integration by parts

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Let $u = \eta, X = \nabla f$ then $\int_M <\nabla f, \nabla \eta> dV = -\int_M \eta div(\nabla f)dV = -\int_M \eta \Delta f dV$

$\int_M \eta \Delta f dV = 0$ for all function η with compact support, this implies

$\Delta f = 0$