§ Laplacian operator 的幾何意義

考慮原點的鄰域(-h,h)， $u: R \to R$

$$u(x) = u(0) + u'(0)x + \frac{x^2}{2}u''(0) + \frac{x^3}{6}u'''(0) + o(x^4)$$

$$\bar{u}(h) = \frac{1}{2h}\int_{-h}^{h} u(x)dx = u(0) + \frac{u''(0)}{6}h^2 + o(h^4)$$

Then $u''(0) = \frac{6}{h^2}(\bar{u}(h) - u(0)) + o(h^2)$

u 在 O 點附近的平均值與 u(0)的差量就是 u(x)的 Laplacian 的幾何意義。

§ 二維的情形

考慮 $D_h = \{(x,y) \mid x^2 + y^2 \le h^2\}$

u(x,y)在(0,0)展開

$$u(x,y) = u(0,0) + \frac{\partial u}{\partial x}(0,0)x + \frac{\partial u}{\partial y}(0,0)y + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(0,0)x^2 + \frac{\partial^2 u}{\partial x \partial y}(0,0)xy + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(0,0)y^2 + o(r^2)$$

其中 $\iint_{D_h} x dA = \iint_{D_h} y dA = 0$ ， $\iint_{D_h} xy dA = 0$

$\iint_{D_h} x^2 dA = \iint_{D_h} y^2 dA$ 合併計算 $\iint_{D_h} (x^2 + y^2) dA = \int_0^{2\pi}\int_0^h r^2 \cdot r dr d\theta = ... = \frac{\pi h^4}{2}$

,,,

$$\Delta u(0,0) = \frac{8}{h^2}(\bar{u}(h) - u(0,0)) + o(1) \quad o(1) \to 0 \text{ as } h \to 0$$

所以 Laplacian 量化了函數在該點附近平均值與中心點的偏差。