§ 1.6 heat flow method

Parabolic partial differential equations

$$\begin{cases} u_t - ku_{xx} = f(x,t), t>0 \\ u\big|_{t=0} = g(x) \end{cases}$$ is a heat equation on R

The particular solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)g(y)dy + \int_0^t \int_{-\infty}^{\infty} G(x-y,t-s)f(y,s)dyds$$

Theorem 1.6.1

Let M be a compact Riemannian manifold。

Then every homotopy class of closed curves in M contains a geodesic。

Consider the mapping $u: S^1 \times [0,\infty) \to M$ ，u=u(s ,t)

$$\frac{\partial}{\partial t}u^i = \frac{\partial^2}{\partial s^2}u^i + \Gamma^i_{jk}(u(s,t))\frac{\partial u^j}{\partial s}\frac{\partial u^k}{\partial s}, t \ge 0 \cdots 1.6.2$$

($u_t^i = u_{ss}^i + \Gamma^i_{jk} u_s^j u_s^k$ $\cdots$1.6.4 or $u_t - u_{ss} = f(s,t)$ as a heat quation)

$u(s,0) = \gamma(s)$ for $s \in S^1$

For some smooth curve $\gamma : S^1 \to M$ in the given homotopy class。

$$E(u(.,t)) = \frac{1}{2}\int_{S^1} g_{ij}(u(s,t))u_s^i u_s^j$$

$\dfrac{dE(u(.,t))}{dt} = -\int_{S^1} g_{ij}u_t^i u_t^j$ 是 t 的漸減函數，且 $\dfrac{d^2E}{dt^2} = \int_{S^1} 2g_{ij}u_{st}^i u_{st}^j \ge 0$

(在 normal coordinates 下計算)，

Thus, the energy $E(u(\cdot,t))$ is a convex function of t, and since we already know that $\frac{d}{dt}E(u(\cdot,t_n)) \to 0$ for some sequence $t_n \to \infty$, we conclude that $\frac{d}{dt}E(u(\cdot,t)) \to 0$ for $t \to \infty$. Thus, again invoking our pointwise estimates, $u_t(s,t) \to 0$ for $t \to \infty$. This implies that $u(s) = \lim_{t\to\infty} u(s,t)$ exists and is geodesic.

This completes the proof. □

在 p.23 能量 E 的 Euler-Lagrange equation 為 geodesic equation。