§ 2.1 Twist Properties

Definition

Let (V,Ω) be a symplectic vector space，dim V=2n

A subspace $E \subset V$ is said to be Lagrangian if its dimension is n and the symplectic form satisfies $\Omega\big|_{E\times E}=0$

A submanifold P of a symplectic manifold is said to be Lagrangian if the tangent space $T_x P$ is a Lagrangian subspace for all $x \in P$

Then for (TM,Ω)，the subspace $H(\theta),V(\theta)$ are Lagrangian by definition。

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Remark 2.2. More generally, a subspace $S \subset V$ is said to be *isotropic* if $\Omega|_{S\times S} = 0$. Because of the nondegeneracy of Ω, an isotropic subspace has at most dimension n, so Lagrangian subspaces are maximal isotropic subspaces.

Exercise 2.3. Let V be a $2n$ dimensional linear space and Ω a symplectic form on V. Prove that there exists a basis $\{e_i, e_{n+i}\}_{1\le i\le n}$ of V such that $\Omega(e_i, e_j) = \Omega(e_{n+i}, e_{n+j}) = 0$ and $\Omega(e_i, e_{n+j}) = \delta_{ij}$ for $1 \le i, j \le n$.

The basis of the Exercise 2.3 gives a decomposition of V as the direct sum of two Lagrangian subspaces.

Exercise 2.4. A subspace $E \subset T_\theta TM$ is Lagrangian iff $J_\theta E = E^\perp$.

Exercise 2.5. Let V be a $2n$ dimensional real vector space, and let Ω be a nondegenerate two-form in V. Define an action of $GL(2n, \mathbb{R})$ on the set of nondegenerate two-forms on V by $(a\Omega)(v, w) = \Omega(av, aw)$ for all vectors v and w in V. Using Exercise 2.3 prove that $GL(2n, \mathbb{R})$ acts transitively on the set of nondegenerate two-forms on V.

Exercise 2.5. Let V be a $2n$ dimensional real vector space, and let Ω be a nondegenerate two-form in V. Define an action of $GL(2n, \mathbb{R})$ on the set of nondegenerate two-forms on V by $(a\Omega)(v, w) = \Omega(av, aw)$ for all vectors v and w in V. Using Exercise 2.3 prove that $GL(2n, \mathbb{R})$ acts transitively on the set of nondegenerate two-forms on V.

Exercise 2.6. Let $V = \mathbb{C}^n$, regarded as a $2n$ dimensional real vector space, and define a two-form Ω on V by $\Omega(v, w) = \operatorname{Re}(\langle Jv, w\rangle)$, where $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ are arbitrary elements of $\mathbb{C}^n$,

$$J(v_1, \ldots, v_n) = (iv_1, \ldots, iv_n)$$

and $\langle v, w\rangle = \sum_{i=1}^n v_k \bar{w}_k$. Prove

1. Ω is a nondegenerate two-form;
2. if $g(v, w) = \operatorname{Re}(\langle v, w\rangle)$ is the usual inner product on $\mathbb{R}^{2n} = \mathbb{C}^n$, then $\Omega(v, w) = g(Jv, w)$;
3. if $E \subset \mathbb{C}^n$ is a real subspace with real dimension n, then E is Lagrangian if and only if $\langle v, w\rangle \in \mathbb{R}$ for all v and w in E;
4. if $E \subset \mathbb{C}^n$ is a Lagrangian subspace, then $a(E)$ is a Lagrangian subspace for all a in the unitary group $U(n)$;
5. $U(n)$ acts transitively on the set of Lagrangian subspaces of $\mathbb{C}^n$.

Exercise 2.7. Let $H := \{a \in GL(2n, \mathbb{R}) : a\Omega = a\}$. Prove that an element $a \in GL(2n, \mathbb{R})$ lies in H if and only if $a^* Ja = J$, where a^* denotes the real transpose of the transformation a relative to the inner product g on $\mathbb{R}^{2n}$. Show that $U(n)$ is a real subgroup of H of dimension n^2.

Lemma 2.8. *Let N be a submanifold of M and*

$$TN^{\perp} := \{(x, v) \in TM \ : \ x \in N, \ v \perp T_x N\}$$

its normal subbundle. The submanifold $TN^{\perp}$ is Lagrangian.

Proof

Exercise 2.9. Given $x \in N$ and $v \in T_x N^\perp$, the shape operator of N at v is the symmetric linear map

$$A_v : T_x N \to T_x N$$

defined as follows. Let V be a C^∞ normal vector field in a neighborhood of x in N with $V(x) = v$. Given $w \in T_x N$, $A_v(w)$ is the orthogonal projection of $\nabla_w V$ onto $T_x N$ (see for example [Sa]). Show that $T_{(x,v)} T N^\perp$ is given by the set of vectors $\xi \in T_{(x,v)} TM$ such that $d_{(x,v)}\pi(\xi) \in T_x N$ and $K_{(x,v)}(\xi) - A_v(d_{(x,v)}\pi(\xi)) \in T_x N^\perp$. Use the above, together with the fact that A_v is a symmetric linear map, to give another proof of the last lemma.

Proposition 2.11 (twist property of the vertical subbundle). *Let E be a Lagrangian subspace of $\subset T_\theta TM$. The subset given by*

$$\{t \in \mathbb{R} : \; d_\theta \phi_t(E) \cap V(\phi_t(\theta)) \neq \{0\}\}$$

is discrete.

Proof