§ 1.5 Jacobi fields and the differential of geodesic flow

J is a Jacobi field along γ_θ $\Leftrightarrow$ J satisfies $\ddot{J}+R(\dot{\gamma}_\theta, J)\dot{\gamma}_\theta = 0$

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

Let $\xi \in T_\theta TM$ and $z : (-\varepsilon, \varepsilon) \to TM$ be an adapted curve to ξ. Then the map $(s,t) \mapsto \pi \circ \phi_t(z(s))$ gives rise to a variation of $\gamma_\theta = \pi \circ \phi_t(\theta)$. The curves $t \mapsto \pi \circ \phi_t(z(s))$ are geodesics and therefore the corresponding variational vector field $J_\xi(t) := \frac{\partial}{\partial s}\big|_{s=0} \pi \circ \phi_t(z(s))$ is a Jacobi vector field with initial conditions given by

$$\begin{cases} J_\xi(0) &= \frac{\partial}{\partial s}\big|_{s=0} \pi \circ \phi_t(z(s)) = d_\theta \pi(\xi) \\ \dot{J}_\xi(0) &= \frac{D}{\partial t}\big|_{t=0} \frac{\partial}{\partial s}\big|_{s=0} \pi \circ \phi_t(z(s)) \\ &= \frac{D}{\partial s}\big|_{s=0} \frac{\partial}{\partial t}\big|_{t=0} \pi \circ \phi_t(z(s)) = \frac{D}{\partial s}\big|_{s=0} Z(s) = K_\theta(\xi). \end{cases}$$

Let us denote by $J(\gamma_\theta)$ the vector space of all solutions of the Jacobi equation (1.3). It is a $2 \dim M$ dimensional vector space. Let us consider the map $i_\theta : T_\theta TM \to J(\gamma_\theta)$ given by

$$i_\theta(\xi) = J_\xi.$$

It is obvious that i_θ is a linear isomorphism.

We say that a Jacobi field is normal to the geodesic γ_θ if $<J(t), \gamma_\theta(t)>=0$ for all t

Lemma

Given $\theta \in TM, \xi \in T_\theta TM$ and $t \in R$ we have

$$d_\theta \phi_t(\xi) = (J_\xi(t), \dot{J}_\xi(t))$$

Proof

$$\begin{aligned} J_\xi(t) &= \frac{\partial}{\partial s}\big|_{s=0} (\pi \circ \phi_t(z(s))) = d_\theta(\pi \circ \phi_t)(\xi) = d_{\phi_t(\theta)}\pi \circ d_\theta \phi_t(\xi); \\ \dot{J}_\xi(t) &= \frac{D}{\partial t} \frac{\partial}{\partial s}\big|_{s=0} \pi \circ \phi_t(z(s)) = \frac{D}{\partial s}\big|_{s=0} \frac{\partial}{\partial t} \pi \circ \phi_t(z(s)) \\ &= \frac{D}{\partial s}\big|_{s=0} \phi_t(z(s)) = K_{\phi_t(\theta)}(d\phi_t(\xi)). \end{aligned}$$

□

By means of this identification we can write

$$\Omega_{\phi_t\theta}(d_\theta\phi_t(\xi), d_\theta\phi_t(\eta)) = \langle -\dot{J}_\xi(t), J_\eta(t)\rangle + \langle J_\xi(t), \dot{J}_\eta(t)\rangle .$$

Since Ω is invariant under ϕ_t, the right hand side should be a constant function of t. This can be checked by differentiation and using the Jacobi equation:

$$\begin{aligned}(\langle -\dot{J}_\xi, J_\eta\rangle + \langle J_\xi, \dot{J}_\eta\rangle)' &= -\langle \dot{J}_\xi, \dot{J}_\eta\rangle - \langle \ddot{J}_\xi, J_\eta\rangle + \langle \dot{J}_\xi, \dot{J}_\eta\rangle + \langle J_\xi, \ddot{J}_\eta\rangle \\ &= \langle J_\eta, R(\dot{\gamma}, J_\xi)\dot{\gamma}\rangle - \langle J_\xi, R(\dot{\gamma}, J_\eta)\dot{\gamma}\rangle = 0.\end{aligned}$$

Exercise 1.41. Show that $\phi_t : TM \to TM$ is an isometry of the Sasaki metric for all $t \in \mathbb{R}$ if and only if M has constant sectional curvature 1.