§ 1.4 The cotangent bundle T^*M

TM 有 symplectic structure、SM 有 contact structure 其定義皆由 M 的 Riemannian metric 決定。Cotangent bundle T^*M 則有內稟的 symplectic structure，亦即不依靠任何 metric。

Let $\tilde{\pi}: T^*M \to M$ be the canonical projection

Given $(x,p) \in T^*M, \xi \in T^*_{(x,p)}M$ 、we define

$A_{(x,p)}(\xi) := p(d_{(x,p)}\tilde{\pi}(\xi))$ is a canonical 0ne-form in T^*M

The symplectic form in T^*M is $\omega = -dA$

Definition

Let M be a Riemannian manifold。The musical isomorphisms

$TM \xrightarrow{b} T^*M, T^*M \xrightarrow{\#} TM$ are defined as

$v^b(\bullet) := < v, \bullet >, < v, u^\# > := u(v)$

Lemma

There exist the following relations:

1. $\alpha = b^*A$
2. $\Omega = b^*\omega$

Where b is the notation that lower indices。

Proof. **Let $\theta = (x, v)$ and $\flat(\theta) = (x, p)$ be points in TM and T^*M respectively. Then,**

$$\begin{aligned} (\flat^* A)_\theta(\xi) &= A(d_\theta \flat(\xi)) = p(d_{\flat(\theta)}\bar{\pi} \circ d_\theta \flat(\xi)) \\ &= p(d_\theta \pi(\xi)) = \langle d_\theta \pi(\xi), v \rangle = \alpha_\theta(\xi), \end{aligned}$$

where in the third equality we used that $\bar{\pi} \circ \flat = \pi$.

Finally, using that the exterior derivative commutes with the pullback, we obtain

$$\flat^*\omega = \flat^*(-dA) = -d(\flat^* A) = -d\alpha = \Omega,$$

concluding the proof of 2. □