§ Contact forms

$\alpha_\theta(\xi) := \ll \xi, G(\theta) \gg = < d_\theta \pi(\xi), v >_x$ is an one-form of TM

Proposition

$\Omega = -d\alpha$

Lemma

Let $\nabla /$ denote the Riemannian connection of the Sasaki metric。Then for all $\eta \in H(\theta)$ we have that $\nabla /_\eta G \in V(\theta)$

**Lemma 1.25.** *Let* $\nabla\!\!/$ *denote the Riemannian connection of the Sasaki metric. Then for all* $\eta \in H(\theta)$ *we have that* $\nabla\!\!/_\eta G \in V(\theta)$.

*Proof.* Let $\{E_1, \ldots, E_n\}$ be an orthonormal frame field that is geodesic at $x = \pi(\theta)$ and defined in a neighborhood $U$ of $x$. This means $\nabla_{E_i} E_j(x) = 0$, for all $i$ and $j$. Let $X_i(y, w) := (E_i(y), 0)$ be the horizontal lift of the vector field $E_i$, in other words, $X_i(y, w) = L_{(y,w)}(E_i(y))$. The vector fields $\{X_1, \ldots, X_n\}$ are orthonormal relative to the Sasaki metric and they span the horizontal subbundle (where defined) in $TM$. Note that it suffices to show that $\nabla\!\!/_{X_j} G$ belongs to $V(\theta)$ for all $j$. The geodesic vector field can be written as

$$G(y, w) = \sum_{i=1}^{n} \langle E_i(y), w \rangle X_i,$$

hence

$$\nabla\!\!/_{X_j} G = \underbrace{\sum_{i=1}^{n} X_j \langle E_i(y), w \rangle X_i}_{(i)} + \underbrace{\sum_{i=1}^{n} \langle E_i(y), w \rangle \nabla\!\!/_{X_j} X_i}_{(ii)} .$$

Since $\pi : TM \to M$ is a Riemannian submersion, the horizontal component of $\nabla\!\!/_{X_j} X_i(\theta)$ equals $\nabla_{E_j} E_i(x) = 0$, therefore the expression $(ii)$ is vertical. To complete the proof of the lemma we need to show that $(i)$ is vertical. In fact we shall prove that it vanishes.

Let $\alpha_j : (-\varepsilon, \varepsilon) \to M$ be a curve adapted to $E_j(x)$ and let $Z_j$ be the parallel transport of $v$ along $\alpha_j$. We have

$$X_j \langle E_i(\cdot), \cdot \rangle = \left.\frac{d}{dt}\right|_{t=0} \langle E_i(\alpha_j), Z_j \rangle = \langle \nabla_{\dot\alpha_j} E_i, v \rangle = 0. \qquad \square$$

*Proof of Proposition 1.24.* Recall that we can write

$$d\alpha\,(\xi_1, \xi_2) = \xi_1\alpha(\xi_2) - \xi_2\alpha(\xi_1) - \alpha\,([\xi_1, \xi_2])\,.$$

Using the definition of $\alpha$ and the symmetry of the connection $\nabla\!\!\!/$ we obtain,

$$d\alpha(\xi_1, \xi_2) = \xi_1 \langle\langle \xi_2, G\rangle\rangle - \xi_2 \langle\langle \xi_1, G\rangle\rangle - \langle\langle [\xi_1, \xi_2], G\rangle\rangle \qquad (1.1)$$

$$= \left\langle\!\left\langle \xi_2, \nabla\!\!\!/_{\xi_1} G\right\rangle\!\right\rangle - \left\langle\!\left\langle \xi_1, \nabla\!\!\!/_{\xi_2} G\right\rangle\!\right\rangle. \qquad (1.2)$$

We keep the notation of the previous lemma. Let us consider the vector fields $Y_i(y, w) := J_{(y,w)}(X_i(y, w))$. For each $(y, w)$, the vector fields

$$\{X_1, \dots, X_n, Y_1, \dots, Y_n\}$$

form an orthonormal basis of $T_{(y,w)}TM$. Note that $[Y_i, Y_j]$ is vertical since each $Y_i$ is tangent to the fibres of $TM$. Therefore using equation (1.1) and the fact that $G$ is horizontal we get

$$d\alpha_\theta|_{V(\theta)\times V(\theta)} = 0.$$

From the lemma and equation (1.2) we obtain

$$d\alpha_\theta|_{H(\theta)\times H(\theta)} = 0.$$

To end the proof it suffices to show that, for all $i$ and $j$, we have

$$d\alpha_\theta(X_i, Y_j) = -\Omega_\theta(X_i, Y_j).$$