§ 1.31 The geometry of tangent bundle

切叢上的向量場是微分幾何、大域分析、數學物理的基本概念。

A spray is a vector field H on the tangent bundle TM that…

And what is a geodesic spray？

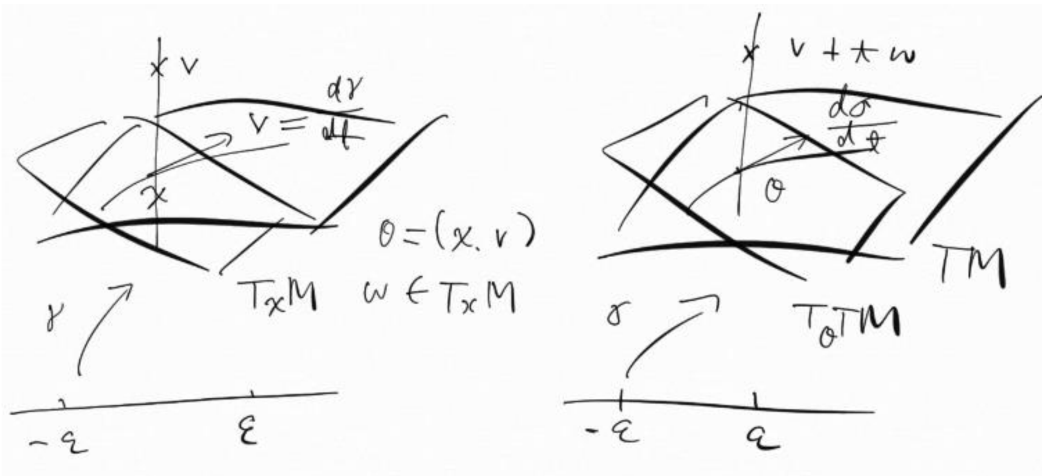
Vector fields on a tangent bundle can be considered as an underlying geometric structure for the theory of second-order differential equations。

The semispray theory has been used in the calculus of variation on manifolds to characterize extremum curves of a variational functional as integral curves of the Hamilton or Euler-Lagrange vector fields。

Sprays and semisprays also provide a natural framework for extension of classical results of analytical mechanics to contemporary mechanical problems and stimulate a broad research in the global theory of nonconservative systems，symmetries，and the constraint theory。

$T_\theta TM = H(\theta) \oplus V(\theta)$ ，我們在 TM 上定義一個 metric 使得 $H(\theta)$ 與 $V(\theta)$ 互相垂直，稱為 Sasaki metric。

建立 TTM 水平與垂直的經典(canonical)子切叢，在 TM 上取 Sasaki metric。



$\pi : TM \to M$ ，$\theta = (x, v) \in TM$

$\pi(\theta) = x$ (projection) ，∇ is the Levi-Civita connection on M given by the metric g。

Using π and ∇，we decompose $T_\theta TM$ into vertical and horizontal subbundles。

切叢 TM 在 $\theta = (x, v) \in TM$ 的切空間分解成垂直與水平兩個子切叢。

Then $d_\theta \pi : T_\theta TM \to TM$

And $V(\theta) = \ker(d_\theta \pi)$ is the tangent space to the fiber T_xM at the point θ

$H(\theta)=\ker K_\theta$，where $K_\theta : T_\theta TM \to T_x M$ is a linear map defined in terms of ∇

Then $T_\theta TM = H(\theta) \oplus V(\theta)$

and we get an isomorphism $T_\theta TM \to T_x M \times T_x M$ given by $\xi \to (d_\theta \pi(\xi)), K_\theta(\xi))$

Finally we define the form ω on TM by setting

$\omega_\theta(\xi,\eta) = < K_\theta(\xi), d_\theta \pi(\eta) > - < d_\theta \pi(\xi), K_\theta(\eta) >$ for all $\xi,\eta \in T_\theta TM$

Then ω is skew-symmetric，nondegenerate and exact，and hence closed，so (TM,ω) is a symplectic manifold。

Theorem

We define the function $H : TM \to R$ by $H(x,v) = \frac{1}{2} < v, v >_x$，and let G denote the vector field on TM obtained from the geodesic flow，then G is the Hamiltonian vector field for H。

That is $dH(\varsigma) = \omega(\varsigma, G)$ for all $\varsigma \in TTM$

Thus，the geodesic flow is the Hamiltonian flow of H on (TM,ω)，so it makes sense to speak of the integrability or nonintegrability of the geodesic flow on a Riemannian manifold。

Examples of manifolds with integrable geodesic flow

§ The classical examples

1. Flat metric on R^n, T^n
2. Surfaces of revolution
3. Left-invariant metric on SO(3)
4. n-dimensional ellipsoids with different principal axes

§ recent examples

Thimm method

Theorem 3.2 (Thimm). *The following manifolds admit Riemannian metrics with integrable geodesic flows:*

- *Real and complex Grassmannians*
- *Distance spheres in* $\mathbb{C}P^{n+1}$
- $SU(n+1)/SO(n+1)$

- $SO(n+1)/SO(n-1)$
- $\mathbb{C}H^n = U(n,1)/U(n) \times U(1)$
- $U(n+1)/O(n+1)$, *which can be viewed as the Lagrangian subspaces of a symplectic vector space*

後面有很多內容 略

Conjecture

Let M be a compact Riemannian manifold whose geodesic flow is integrable。
Then $\pi_1(M)$ has polynomial growth。Moreover，if $\pi_1(M)$ is finite，then M is rationally elliptic。

Conjecture

Let M be a Riemannian manifold with integrable geodesic flow such that the integrals are real analytic，then the topological entropy of the geodesic flow is zero。

Q：

Suppose that M admits metrics with integrable geodesic flow。
Can we classify such metric？

§ Connection map K

Let $\xi \in T_\theta TM$

$z:(-\varepsilon,\varepsilon)\to TM$ be a curve with $\begin{cases} z(0)=\theta \\ \dot{z}(0)=\xi \end{cases}$

which rise a curve $\alpha:(-\varepsilon,\varepsilon)\to M$，with $\alpha := \pi_0 z$，and a vector field Z along α equivalently $z(t)=(\alpha(t),Z(t))$

Def $K:TTM\to TM$ by $K_\theta(\xi):=(\nabla_\alpha Z)(0)$ and $H(\theta):=\ker K_\theta$

Lemma

1. K_θ is well defined
2. K_θ is linear

另一種說法是

$TTM = H\oplus V$，其中 $V=\pi^* TM$，$H=\ker(\pi^*\nabla_{\cdot}\xi)$，$\xi\in\chi_{TM}$

And $\pi^*\nabla_{\pi^* X}\xi = \pi^* X$

A Riemannian metric on TM is a (smoothly varying) choice of inner product on the double tangent space T_vTM for each $v\in TM$. Since $\pi : TM\to M$ is a vector bundle over M, each T_vTM has as a subspace the vertical tangent space V_vTM, which consists of the velocity vectors of curves in the vector space $T_{\pi(v)}M$, and thus can be canonically identified with T_vM. The Levi-Civita connection of (M,g) provides a canonical horizontal subspace H_vTM, which consists of the velocity vectors of curves $(\gamma(t),V(t))\in TM$ such that $v=(\gamma(0),V(0))$ and $\nabla_{\dot\gamma}V=0$.

The upshot of all this is that we have a direct sum decomposition $TTM = VTM \oplus HTM$, with canonical isomorphisms $V_vTM \simeq T_{\pi(v)}M$ (described earlier) and $H_vTM \simeq T_{\pi(v)}M$ (by sending the velocity of (γ, V) to the velocity of γ). If this isn't intuitive, think about the Euclidean case - if you have a tangent vector v to $p = \pi(v) \in \mathbb{R}^n$, then the directions you can move it decouple in to one copy of R^n for the motion of the basepoint and another copy for the motion of the vector.

The Sasaki metric can then be naturally defined by declaring V_vTM and H_vTM to be orthogonal, with the metric on each factor just being the pullback of g from $T_{\pi(v)}M$ via the canonical isomorphisms.

This construction works for any vector bundle E (over a Riemannian manifold M) equipped with a fibre metric and compatible connection: the vertical tangent spaces take the fibre metric from E, while the horizontal spaces (as defined by the connection) take the metric from TM. I have seen this general construction called the *Kaluza-Klein* metric.

Another equivalent way of constructing the horizontal subbundle is by means of the *horizontal lift*

$$L_\theta : T_xM \to T_\theta TM,$$

which is defined as follows ($\theta = (x, v)$). Given $v' \in T_xM$ and $\alpha : (-\varepsilon, \varepsilon) \to M$ an adapted curve to v', let $Z(t)$ be the parallel transport of v along α. Let $\sigma : (-\varepsilon, \varepsilon) \to TM$ be the curve $\sigma(t) = (\alpha(t), Z(t))$. Then

$$L_\theta(v') := \dot{\sigma}(0) \in T_\theta TM.$$

It is immediate from the definition of parallel transport that $K_\theta(L_\theta(v')) = 0$, for all $v' \in T_xM$.

Lemma

1. L_θ is well defined
2. L_θ is linear
3. $\ker(K_\theta) = im(L_\theta)$

From the lemma we conclude that $T_\theta TM = H(\theta) \oplus V(\theta)$
And that the map $j_\theta : T_\theta TM \to T_xM \times T_xM$ given by $j_\theta(\xi) = (d_\theta\pi(\xi), K_\theta(\xi))$
is a linear isomorphism。

用如此分解 $T_\theta TM = H(\theta) \oplus V(\theta)$ 我們在 TM 上定義一個 metric 使得 $H(\theta)$ 與 $V(\theta)$ 互相垂直，稱為 Sasaki metric

$$<< \xi, \eta >> := < d_\theta\pi(\xi), d\theta\pi(\eta) >_{\pi(\theta)} + < K_\theta(\xi), K_\theta(\eta) >_{\pi(\theta)}$$

§ Sasaki metric

If (M , g) is a Riemannian manifold，then we have an associated canonical torsion-free connection ∇ on M。

From now on ∇ is the Levi-Civita connection and R denotes its curvature tensor。

With such connection one defines the Sasaki metric on the manifold TM：

$g := \pi^* g \oplus \pi^* g$

It follows immediately that $\nabla^* g = 0$

(M ,g) is a Riemann manifold，then

$(TM, \tilde{g})$ is a Riemann manifold with Sasaki metric $\tilde{g}$

$d\sigma^2 = g_{ij} dx^i dx^j + g_{ij} Dv^i Dv^j$ where D is the covariant derivative $Dv^i = dv^i + \Gamma^i_{jk} v^j v^k$

In components，$\tilde{g}_{jk} = g_{jk} + g_{\alpha\gamma} \Gamma^{\alpha}_{\mu j} \Gamma^{\gamma}_{\eta k} v^{\mu} v^{\eta}$ [Sasaki metric]

The geodesic vector field $G : TM \to TTM$ is given by

$$G(\theta) := \frac{\partial}{\partial t}\Big|_{t-0} \phi_t(\theta) = \frac{\partial}{\partial t}\Big|_{t=0} (\gamma_\theta(t), \dot{\gamma}_\theta(t))$$

Where γ_θ is，as usual，the unique geodesic with initial condition $\theta = (x, v)$

But，note that $t \to \dot{\gamma}_\theta(t)$ is the parallel transport of v along γ_θ。

Therefore，$G(\theta) = L_\theta(v)$，or equivalently，$G(\theta) = L_\theta(v) = (v, 0)$ using the identification j_θ。