§ 1.2 Symplectic and contact manifolds

在一個辛流形上的哈密頓：

(1)哈密頓流線 (2)哈密頓向量場 (3)流線的可積性

What is the Hamiltonian？

The Hamiltonian H of a system is defined as $H(q,\dot{q},t)=\dot{q}\,p-L(q,\dot{q},t)$ ，

where q is a generalized coordinates ， $p=\dfrac{\partial L}{\partial \dot{q}}$ is a generalized momentum ，L is the Lagrangian 。

The Hamilton equations are the equations for the flow of the vector field X_H satisfying $i(X_H)\omega=-dH$

(與下列差一負號？)

the Hamilton equations： $\begin{cases} \dot{x}^i=\dfrac{\partial H}{\partial p_i} \ldots\text{.(1')} \\ \dot{p}_i=-\dfrac{\partial H}{\partial x^i} \ldots\text{(2')} \end{cases}$

(M,ω) is a smyplectic manifold if

1. ω is a 2-form，
2. $d\omega=0$
3. Nondegenerate ：If $\omega_p(X,Y)=0$ for all $Y\in T_pM$ then X=0

   (當用局部座標系表示 $\omega=\sum_{i,j}\omega_{ij}dx^i\wedge dx^j$ 則 $\det(\omega_{ij})\neq 0$)

$H:M\to R$ a Hamiltonian is a smooth function

X_H Hamiltonian vector field if $\omega(X_H,Y)=dH(Y)$ for all $Y\in T_pM$

(or equivalently $\iota X_H\omega=dH$ ，or simply $dH(v)=\omega(v,\xi_h)$ for all $v\in TM$)

φ_t the flow of X_H is called the Hamiltonian flow of H

Theorem 1.2.1

The function H is an integral of the Hamiltonian phase flow with Hamiltonian H

(The mathematical formulation of the mechanical principle of the conservation of energy。)

Proof

$dH(X_H)=\omega(X_H,X_H)=0$ (因為 ω 是反對稱。)

Lemma $L_{X_H}\omega=0$

By Cartan magic formula

$L_{X_H}\omega=\iota_{X_H}(d\omega)+d(\iota_{X_H}\omega)=0+0=0$

One way to place the geodesic equations of M into the context of Hamiltonian dynamics is to look at the cotangent bundle T^*M 。

We put the canonical symplectic structure on T^*M and define the Hamiltonian H on T^*M in local coordinates by $H(x,p)=\frac{1}{2}\sum g^{ij}p_i p_j$

An equivalent approach is to put a symplectic structure directly on TM，i.e. a closed，nondegenerate，smooth 2-form ω on TTM。

Exercise

Show that $L_{X_H}\omega=0 \Leftrightarrow$ for all $t\in R$ $\varphi_t^*\omega=\omega$ where φ_t is the flow of X_H

(i.e. A Hamiltonian phase flow preserves the symplectic form ω)

Proof

$L_X\omega:=\frac{d}{dt}(\varphi_t^*\omega)\big|_{t=0}=\lim_{t\to 0}\frac{1}{t}(\varphi_t^*\omega-\omega)$

§ Poisson bracket {F,G}

Let F,G be Hamiltonians on the symplectic manifold (M,ω)，and let ξ_G denote the Hamiltonian vector field of G。i.e. $\omega(\xi_G,Y)=dG(Y)$

We define $\{F,G\}=dF(\xi_G)$，the derivative of F in the direction of the Hamiltonian flow of G。

1. Bilinear
2. Skew-symmetric
3. Satisfies the Jacobi identity

So，is a Lie bracket for the algebra of Hamiltonians on M

Two functions f ,g on M are said to be in involution if they Poisson-commute ({f ,g}=0)

From the definition of the Poisson bracket that a function f is an integral for the Hamiltonian flow of H if and only if f and H are in involution。

Definition

Suppose that $\{f_1, f_2, ..., f_n\}$ is a set of integrals of a flow on M。

We say these integrals are independent at $x \in M$ if their differentials $\{df_1, df_2, ..., df_n\}$

at x form a linearly independent subset of $T_x^* M$

Definition

The flow of a Hamiltonian H on a symplectic manifold M is said to be integrable (or completely integrable) if there exist n everywhere independent integrals $f_1 = H, f_2, ..., f_n$ 0f the flow which are in involution。

A classical theorem of Liouville states that when a Hamiltonian flow is integrable, the flow itself is geometrically very simple. Liouville's theorem asserts the existence of *action-angle coordinates* on M in which the flow behaves as quasiperiodic flows on tori. Thus, the phase space of an integrable Hamiltonian system is foliated by invariant tori.

§ Contact manifold 切觸流形

M is a (2n-1)dim orientable manifold

α is a 1-form in M called a contact form $\Leftrightarrow \alpha \wedge (d\alpha)^{n-1}$ never vanishes

(M, α) is called a contact manifold

X is a canonical vector field (called characteristic vector field) if $\iota_X \alpha = 1, \iota_X d\alpha = 0$

(Then $L_X \alpha = d\iota_X \alpha + \iota_X d\alpha = 0$)