§ 1.1 geodesic flow

1. 幾何觀點的 geodesic flow
2. 力學觀點的 geodesic flow：能量 $E(\gamma)=\frac{1}{2}\int g_{jk}(x(t))\dot{x}^j\dot{x}^k\,dt$ 的 Euler-Lagrange flow。

M is a complete Riemannian manifold

$\gamma_{(x,v)}(t)$ is the unique geodesic with $\begin{cases}\gamma_{(x,v)}(0)=x\\ \dot{\gamma}_{(x,v)}(0)=v\end{cases}$

TM is the tangent bundle。

$\phi_t:TM\to TM$，$\phi_t(x,v):=(\gamma_{(x,v)}(t),\dot{\gamma}_{(x,v)}(t))$ is a diffeomorphism

$\phi_{t=0}(x,v)=(x,v)=identity$

Then $\{\phi_t\}$ is a flow，with $\phi_{t+s}=\phi_t\circ\phi_s$

$SM=\{v\,|\,v\in TM,|v|=1\}$

$\because$ geodesic travel with constant speed，ϕ_t leaves SM invariant。
That is，given $(x,v)\in SM$ for all then $\phi_t(x,v)\in SM$。
The restriction of ϕ_t to SM is called the geodesic flow of g。
(質點沿 geodesic 走 不受力 加速度=0 速度是常數。)

§ Euler-Lagrange flows

$L:TM\to R$ Lagrangian，$u:[0,1]\to M$
(任何守恆的力學系統$(M,<.,.>,-dU)$的運動是由 Lagrangian L:=K-U 所決定的作用量的 critical points。)

The action(作用量) of L，$A(u)=\int_0^1 L(u(t),\dot{u}(t))dt$

考慮 A 的變分，u is a critical point $\Leftrightarrow$

$\frac{\partial L}{\partial u}(u,\dot{u})-\frac{d}{dt}(\frac{\partial L}{\partial v}(u,\dot{u}))=0$，$v=\dot{u}$ (Euler-Lagrange equation)

If M is compact，the extremals(critical point) of A give rise to a complete flow $\phi_t:TM\to TM$ called the Euler-Lagrange flow of the Lagrangian。

(Does it mean $\phi_t(x,v)=(u(t),\dot{u}(t))$ ？)

We shall say that a Lagrangian L is convex and superlinear if the following two properties are satisfied。

1. Convexity：

   We require that $L\big|_{T_xM}: T_xM \to R$ has positive definite Hessian for all $x \in M$ 。

   This condition is usually known as Legendre condition。In local coordinates this means that $\dfrac{\partial^2 L}{\partial v^i \partial v^j}$ is positive definite。

2. Superlinearity：

   There exists a Riemannian metric such that $\lim\limits_{|v|\to\infty} \dfrac{L(x,v)}{|v|} = +\infty$ uniformly on x

(M 是 compact set 極值一定存在。)

In classical field theories，the Lagrangian specification of the flow field is a way of looking at fluid motion where the observer follows an individual fluid parcel (包裹)as it moves through space and time。

This can be visualized as sitting in a boat and drifting(漂流)down a river。

The Eulerian specification of the flow field is a way of looking at fluid motion that focuses on specific locations in the space through which the fluid flows as time passes。

This can be visualized by sitting on the bank(岸邊) of a river and watching the water pass the fixed location.

$$u(X(x_0,t),t) = \frac{\partial X}{\partial t}(x_0,t)$$

It is well-known that geodesics can be seen as the solution of the Euler-Lagrange equation of the following convex and superlinear Lagrangian：

$L(x,v) := \dfrac{1}{2} g_x(v,v)$ ，where g denotes the Riemannian metric of M

[Riemannian Geometry and Geometric Analysis] by Jurgen Jost p.23

Lemma

The Euler-Lagrange equations for the energy E are (equations of geodesic。)

$$\ddot{x}^i(t) + \Gamma^i_{jk}(x(t))\,\dot{x}^j(t)\,\dot{x}^k(t) = 0$$

Where $\Gamma^i_{jk} = \dfrac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l})$

1. For $I(x)=\int_a^b f(x(t),\dot{x}(t),t)dt$ then the E-L equation are $\frac{\partial f}{\partial x^i}-\frac{d}{dt}(\frac{\partial f}{\partial \dot{x}^i})=0$

2. $E(\gamma)=\frac{1}{2}\int g_{jk}(x(t))\dot{x}^j\dot{x}^k\,dt$

Note that $\frac{\partial}{\partial \dot{x}^i}(g_{jk}\dot{x}^j\dot{x}^k)=g_{ik}\dot{x}^k+g_{ji}\dot{x}^j$ 因為 $\frac{\partial}{\partial \dot{x}^i}\dot{x}^j=\delta_{ij}$

The E-L equations are $\frac{\partial}{\partial x^i}(g_{jk}\dot{x}^j\dot{x}^k)-\frac{d}{dt}(\frac{\partial}{\partial \dot{x}^i}g_{jk}\dot{x}^j\dot{x}^k)=0$

$$\frac{\partial g_{jk}}{\partial x^i}\dot{x}^j\dot{x}^k-\left\{\frac{\partial g_{ik}}{\partial x^l}\frac{\partial x^l}{\partial t}\dot{x}^k+\frac{\partial g_{jl}}{\partial x^l}\frac{\partial x^l}{\partial t}\dot{x}^j+g_{ik}\ddot{x}^k+g_{ji}\ddot{x}^j\right\}=0$$

$$g_{jk,i}\dot{x}^j\dot{x}^k-\left\{g_{ik,l}\dot{x}^l\dot{x}^k+g_{ji,l}\dot{x}^l\dot{x}^j+g_{ik}\ddot{x}^k+g_{ji}\ddot{x}^j\right\}=0$$

Renaming some indices and using the symmetry $g_{ik}=g_{ki}$, we get

$$2g_{\ell m}\ddot{x}^m+(g_{\ell k,j}+g_{j\ell,k}-g_{jk,\ell})\dot{x}^j\dot{x}^k=0,\quad \ell=1,\dots,d,$$

And from this $g^{il}g_{lm}\ddot{x}^m+\frac{1}{2}g^{il}(g_{lk,j}+g_{jl,k}-g_{jk,l})\dot{x}^j\dot{x}^k=0$

$g^{il}g_{lm}=\delta_{im}$ ，so $g^{il}g_{lm}\ddot{x}^m=\ddot{x}^i$ ，that is $\ddot{x}^i+\Gamma^i_{jk}\dot{x}^j\dot{x}^j=0$

參考 [Lecture on Geometry of Manifolds] by Liviu I. Nicolaescu p.137