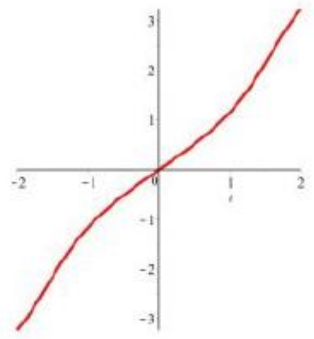
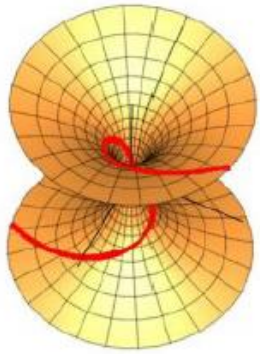
§ Wormhole

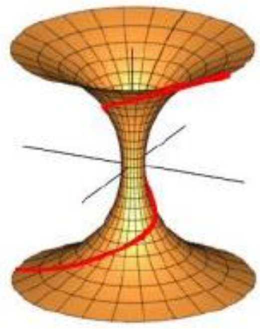
[Differential Geometry in Physics] p.159

Michael Morris and Kip Thorne 1987

預備



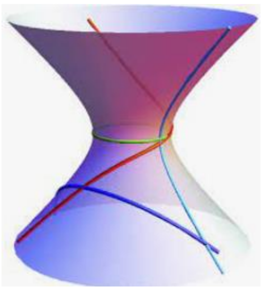




Catenoid

$$\begin{cases} x = a\cosh(\frac{v}{a})\cos u \\ y = a\cosh(\frac{v}{a})\sin u \quad u \in [0, 2\pi) \\ z = v \end{cases}$$

The geodesic of Catenoid is



$$ds^2 = -c^2dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta d\varphi^2)$$

接下來考慮 Cartan structure equations

$$d\omega^i = \sum_j \omega^j \wedge \omega_j^i \text{ ， } \omega_j^k = \sum_i \Gamma_{ij}^k \omega^i$$

$$d\omega_i^j = \omega_i^k \wedge \omega_k^j$$

$$\Omega_k^l(X,Y) = \omega^l(R(X,Y), X_k) \text{ then } \Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j$$

取 coframe $\theta^0 = cdt, \theta^1 = dl, \theta^2 = \sqrt{b_0^2 + l^2}d\theta, \theta^3 = \sqrt{b_0^2 + l^2}\sin\theta d\varphi$

則 $d\theta^0 = d\theta^1 = 0$，接著求 connection form ω

$$d\theta^2 = \frac{l}{\sqrt{b_0^2+l^2}} dl \wedge d\theta = \frac{-l}{\sqrt{b_0^2+l^2}} d\theta \wedge dl = \frac{-l}{b_0^l + l^2} \theta^2 \wedge \theta^1 \ldots(1)$$

$$d\theta^3 = \frac{1}{\sqrt{b_0^2+l^2}} \sin\theta dl \wedge d\varphi + \cos\theta \sqrt{b_0^2+l^2} d\theta \wedge d\varphi$$

$$= \frac{-1}{b_0^2+l^2} \theta^3 \wedge \theta^1 - \frac{\cot\theta}{\sqrt{b_0^2+l^2}} \theta^3 \wedge \theta^2 \ldots(2)$$

$d\theta^3 = \theta^1 \wedge \omega_1^3 + \theta^2 \wedge \omega_2^3 + \theta^3 \wedge \omega_3^3$ (note that $\omega_3^3 = 0$，與(2)比較)

$d\theta^2 = \theta^1 \wedge \omega_1^2 + \theta^3 \wedge \omega_3^2$ ($\theta^3 \wedge \omega_3^2 = 0$，與(1)比較)

算(猜)出 connection form ω

$$\omega_1^2 = \frac{l}{b_0^2+l^2} \theta^2, \omega_1^3 = \frac{l}{b_0^2+l^2} \theta^3, \omega_2^3 = \frac{\cot\theta}{\sqrt{b_0^2+l^2}} \theta^3$$

$$\omega_1^2 = -\omega_2^1, \omega_1^3 = -\omega_3^1, \omega_3^2 = -\omega_2^3$$

$$\Omega^1{}_2 = d\omega^1{}_2 + \omega^2{}_1 \wedge \omega^2{}_1 = -\frac{b_o^2}{(b_o^2+l^2)^2} \theta^1 \wedge \theta^2,$$
$$\Omega^1{}_3 = d\omega^1{}_3 + \omega^1{}_2 \wedge \omega^2{}_3 = -\frac{b_o^2}{(b_o^2+l^2)^2} \theta^1 \wedge \theta^3,$$
$$\Omega^2{}_3 = d\omega^2{}_3 + \omega^2{}_1 \wedge \omega^1{}_3 = \frac{b_o^2}{(b_o^2+l^2)^2} \theta^2 \wedge \theta^3.$$

$$R_{2323} = -R_{1212} = R_{1313} = \frac{b_o^2}{(b_o^2+l^2)^2},$$

The only non-zero component of Ricci tensor is $R_{11} = -2\frac{b_0^2}{(b_0^2+l^2)^2}$

Of course, this space is a 4-dimensional continuum, but since the space is spherically symmetric, we may get a good sense of the geometry by taking a slice with $\theta = \pi/2$ at a fixed value of time. The resulting metric ds_2 for the surface is

$$ds_2{}^2 = dl^2 + (b_o^2 + l^2)\, d\phi^2. \tag{6.61}$$

Let $r^2 = b_o^2 + l^2$. Then $dl^2 = (r^2/l^2)\, dr^2$ and the metric becomes

$$ds_2{}^2 = \frac{r^2}{r^2 - b_o^2} dr^2 + r^2\, d\phi^2, \tag{6.62}$$
$$= \frac{1}{1 - \frac{b_o^2}{r^2}} dr^2 + r^2\, d\phi^2. \tag{6.63}$$

It is a Catenoid of revolution，so the equation of geodesics are given by

$$\varphi = \pm c\int \frac{\sqrt{1+f'^2}}{r\sqrt{r^2-c^2}}\,dr \quad \text{with} \quad f(r) = b_0 \cosh^{-1}(\frac{r}{b_0})$$

and we get $\varphi = \pm c\int \frac{1}{\sqrt{r^2-b_0^2}\sqrt{r^2-c^2}}\,dr$

[Ellis Wormhole] H. G. Ellis and K. A. Bronnikov 1969
[minimal surface]