

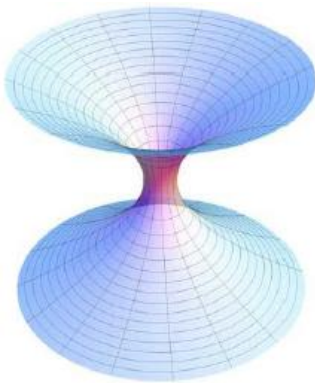
§ Morris-Thorne Wormhole 1988

$$\text{Catenoid} \begin{cases} x = a \cosh\left(\frac{v}{a}\right) \cos u \\ y = a \cosh\left(\frac{v}{a}\right) \sin u, 0 \leq u \leq 2\pi \\ z = v \end{cases}$$

$$ds^2 = -e^{2\phi} dt^2 + \left(1 - \frac{b}{r}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

$\phi(r)$: redshift function ◦ Change in frequency of electromagnetic radiation in gravitation field ◦

$b(r)$: shape function



properties of the metric :

1. Spherically symmetric and static
2. Radial coordinate r such that circumference of circle centered around throat given by $2\pi r$
3. r decreases from $+\infty$ to $b = b_0$ (minimum radius) at throat , then increases from b_0 to $+\infty$
4. At throat exists coordinate singularity where r component diverges
5. Proper radial distance/ r runs from $-\infty$ to $+\infty$ and

vice versa

$$G_{ij} = 8\pi G T_{ij}$$

Cartan's Structure equations

$$d\omega^i = \omega^j \wedge \omega_j^i, \quad \omega_j^k = \sum_i \Gamma_{ij}^k \omega^i$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

$$\omega^0 = -e^\phi dt, \omega^1 = \left(1 - \frac{b}{r}\right)^{-\frac{1}{2}} dr, \omega^2 = r d\theta, \omega^3 = r \sin \theta d\phi$$

$$d\omega^0 = -e^\phi \phi' dr \wedge dt = \phi' e^\phi dt \wedge dr$$

$$d\omega^1 = 0$$

$$d\omega^2 = dr \wedge d\theta$$

$$d\omega^3 = \sin \theta dr \wedge d\phi + r \cos \theta d\theta \wedge d\phi$$

$$dt = -\frac{1}{e^\phi} \omega^0, \quad dr = \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} \omega^1, \quad d\theta = \frac{1}{r} \omega^2, \quad d\phi = \frac{1}{r \sin \theta} \omega^3$$

$$d\omega^0 = \omega^1 \wedge \omega_1^0 + \omega^2 \wedge \omega_2^0 + \omega^3 \wedge \omega_3^0 = -\phi' e^\phi dr \wedge dt$$

$$\Rightarrow \omega_1^0 = -\phi' e^\phi \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} dt, \omega_0^1 = \phi' e^\phi \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} dt$$

$$\text{同理 } \omega_1^2 = \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} d\theta, \omega_1^3 = \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} \sin \theta d\phi, \omega_2^3 = \cos \theta d\phi$$

$$\text{例如 } d\omega^3 = \omega^0 \wedge \omega_0^3 + \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = \sin \theta dr \wedge d\phi + r \cos \theta dr \wedge d\phi$$

$$\sin \theta dr \wedge d\phi = \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} dr \wedge \omega_1^3, r \cos \theta d\theta \wedge d\phi = r d\theta \wedge \omega_2^3$$

$$\text{得 } \omega_1^3 = \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} \sin \theta d\phi, \omega_2^3 = \cos \theta d\phi$$

$$\omega_i^j = \begin{bmatrix} 0 & -\phi' e^\phi \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} dt & 0 & 0 \\ \phi' e^\phi \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} dt & 0 & -\left(1 - \frac{b}{r}\right)^{\frac{1}{2}} d\theta & -\sin \theta \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} d\phi \\ 0 & \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} d\theta & 0 & -\cos \theta d\phi \\ 0 & \sin \theta \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} d\phi & \cos \theta d\phi & 0 \end{bmatrix}$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = -\frac{1}{2} R_{mni}^j \omega^m \wedge \omega^n$$

$$\text{例如 } \Omega_3^2 = d\omega_3^2 - (\omega_3^0 \wedge \omega_0^2 + \omega_3^1 \wedge \omega_1^2 + \omega_3^3 \wedge \omega_2^2 + \omega_3^3 \wedge \omega_2^3)$$

$$= d(-\cos \theta d\phi) - (-\sin \theta \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} d\phi \wedge \left(1 - \frac{b}{r}\right)^{\frac{1}{2}} d\theta) = \frac{b}{r} \sin \theta d\theta \wedge d\phi$$

$$= \frac{b}{r} \sin \theta \times \frac{1}{r} \times \frac{1}{r \sin \theta} \omega^2 \wedge \omega^3 = \frac{b}{r^3} \omega^2 \wedge \omega^3$$

$$R_{323}^2 = \Omega_3^2(E_2, E_3) = \frac{b}{r^3}$$

$$R_{101}^0 = \left(1 - \frac{b}{r}\right) (-\phi'' - (\phi')^2) - \frac{\phi'(b'r - b)}{2r^2}$$

$$R_{212}^1 = R_{121}^2 = -R_{112}^2 = -R_{221}^1 = \frac{b'r - b}{2r^3}$$

$$R_{220}^0 = R_{020}^2 = -R_{202}^0 = -R_{002}^2 = \frac{\phi'(1 - \frac{b}{r})}{r}$$

$$R_{131}^3 = R_{313}^1 = -R_{331}^1 = -R_{113}^3 = \frac{b'r - b}{2r^3}$$

$$R_{330}^0 = R_{030}^3 = -R_{303}^0 = -R_{003}^3 = \frac{\phi'(1-\frac{b}{r})}{r}$$

$$R_{232}^3 = R_{323}^2 = -R_{223}^3 = -R_{332}^2 = \frac{b}{r^3}$$

$$R_{ji}^j = R_{jij}^i = -R_{ijj}^j = -R_{jji}^i$$

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{jilk}$$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

Ricci curvature tensor

$$R_{ij} = \sum_k R_{ikj}^k$$

$$R_{00} = R_{101}^0 + R_{200}^2 + R_{030}^3 = (1-\frac{b}{r})(\phi'' + (\phi')^2) + \frac{\phi'(b'r-b)}{2r^2} + \frac{2\phi'(1-\frac{b}{r})}{r}$$

$$R_{11} = R_{101}^0 + R_{111}^1 + R_{121}^2 + R_{131}^3 = (1-\frac{b}{r})(-\phi'' - (\phi')^2) - \frac{\phi'(b'r-b)}{2r^2} + \frac{b'r-b}{r^3}$$

$$R_{22} = R_{202}^0 + R_{121}^2 + R_{232}^3 = \frac{-\phi'(1-\frac{b}{r})}{r} + \frac{b'r-b}{2r^3} + \frac{b}{r^3} = R_{33}$$

Curvature scalar

$$R = -R_{00} + R_{11} + R_{22} + R_{33}$$

$$= 2(1-\frac{b}{r})(-\phi'' - (\phi')^2) - \frac{\phi'(b'r-b)}{r^2} - \frac{4\phi'(1-\frac{b}{r})}{r} + \frac{2b'r}{r^3}$$

Einstein Tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$, $R_{\mu\nu}$ is Ricci tensor , R is scalar curvature

Then $G_{00} =$

$$G_{11} =$$

$$G_{22} =$$

$$G_{33} =$$

$$G_{tt} = R_{tt} - \frac{1}{2} R g_{tt} = \frac{b'}{r^2}$$

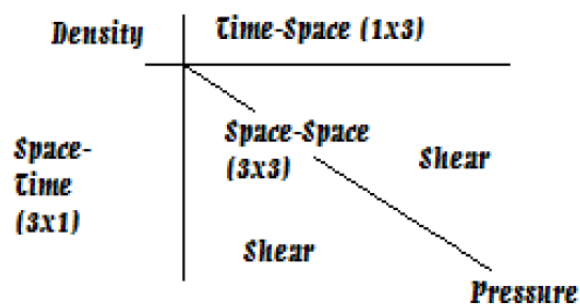
$$G_{rr} = R_{rr} - \frac{1}{2} R g_{rr} = -\frac{b}{r^3} + \frac{2\Phi'}{r} \left(1 - \frac{b}{r}\right)$$

$$G_{\theta\theta} = R_{\theta\theta} - \frac{1}{2} R g_{\theta\theta} = \left(1 - \frac{b}{r}\right) \left[\Phi'' - \frac{b'r-b}{2r(r-b)} \Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)} \right] = G_{\phi\phi}$$

$$G_{\phi\phi} = R_{\phi\phi} - \frac{1}{2} R g_{\phi\phi} = \left(1 - \frac{b}{r}\right) \left[\Phi'' - \frac{b'r-b}{2r(r-b)} \Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)} \right] = G_{\theta\theta}$$

Stress-Energy Tensor

$$T_{\alpha\beta} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$



Equation of State

Energy density $\rho = \frac{b'}{8\pi r^2}$

Tension $\tau = \frac{1}{8\pi r^2} \left(\frac{b}{r} - 2(r-b)\phi' \right)$

Pressure(stress) $p = \frac{r}{2} |(\rho - \tau)\phi' - \tau| - \tau$

Wormhole Embedding Diagram