§ Morris-Thorne Wormhole 1987

Catenoid $\begin{cases} x = a\cosh(\frac{v}{a})\cos u \\ y = a\cosh(\frac{v}{a})\sin u \,, 0 \le u \le 2\pi \\ z = v \end{cases}$

The MT metric for this spherically symmetric geometry is

$ds^2 = -c^2dt^2 + dl^2 + (b_0^2 + l^2)(d\theta^2 + \sin^2\theta d\phi^2)$ ，Where b_0 is a constant。

$$d\omega^i = \omega^j \wedge \omega_j^i$$

$$\omega_i^j + \omega_j^i = 0$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j$$

$$\omega^0 = cdt, \omega^1 = dl, \omega^2 = \sqrt{b_0^2 + l^2}\, d\theta, \omega^3 = \sqrt{b_0^2 + l^2}\sin\theta d\phi$$

$$d\omega^0 = d\omega^1 = 0$$

$$d\omega^2 = \frac{l}{\sqrt{b_0^2 + l^2}} dl \wedge d\theta = \omega^1 \wedge \omega_1^2 + \omega^3 \wedge \omega_3^2 \text{ ， } \therefore \omega_1^2 = \frac{l}{\sqrt{b_0^2 + l^2}} d\theta$$

$$d\omega^3 = \omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = dl \wedge \omega_1^3 + \sqrt{b_0^2 + l^2}\, d\theta \wedge \omega_2^3$$

$$= \frac{l\sin\theta}{\sqrt{b_0^2 + l^2}} dl \wedge d\phi + \sqrt{b_0^2 + l^2}\cos\theta d\theta \wedge \phi$$

$$\therefore \omega_1^3 = \frac{l\sin\theta}{\sqrt{b_0^2 + l^2}} d\phi, \omega_2^3 = \cos\theta d\phi$$

$$\Omega_1^2 = d\omega_1^2 - \omega_1^3 \wedge \omega_3^2 = d(\frac{l}{\sqrt{b_0^2 + l^2}} d\theta) - (\frac{l}{b_0^2 + l^2}\omega^3) \wedge (-\frac{\cot\theta}{\sqrt{b_0^2 + l^2}}\omega^3)$$

$$= \frac{b_0^2}{(b_0^2 + l^2)\sqrt{b_0^2 + l^2}} dl \wedge d\theta = \frac{b_0^2}{(b_0^2 + l^2)^2}\omega^1 \wedge \omega^2$$

$$\Omega_1^3 = d\omega_1^3 - \omega_1^2 \wedge \omega_2^3 = \ldots = \frac{b_0^2}{(b_0^2 + l^2)^2}\omega^1 \wedge \omega^3$$

$$\Omega_2^3 = d\omega_2^3 - \omega_2^1 \wedge \omega_1^3 = \ldots = -\frac{b_0^2}{(b_0^2 + l^2)^2} \omega^2 \wedge \omega^3$$

$$R_{121}^2 = \Omega_1^2(E_1, E_2) = \frac{b_0^2}{(b_0^2 + l^2)^2}$$

$$R_{232}^3 = \Omega_2^3(E_2, E_3) = -\frac{b_0^2}{(b_0^2 + l^2)^2}$$

$$R_{131}^3 = \Omega_1^3(E_1, E_3) = \frac{b_0^2}{(b_0^2 + l^2)^2}$$

$$R_{11} = R_{211}^2 + R_{311}^3 = -R_{121}^2 - R_{131}^3 = \frac{-2b_0^2}{(b_0^2 + l^2)^2}$$

$$R_{22} = R_{122}^1 + R_{322}^3 = R_{121}^2 + R_{322}^3 = 0$$

$$R_{33} = R_{133}^1 + R_{233}^2 = -R_{133}^1 - R_{232}^3 = 0$$

在一固定時間，取 $\theta = \frac{\pi}{2}$ 得 $ds^2 = dl^2 + (b_0^2 + l^2) d\phi^2$

令 $r^2 = b_0^2 + l^2$ 則 $dl^2 = (\frac{r^2}{l^2}) dr^2$

$ds^2 = \frac{1}{1 - (\frac{b_0}{r})^2} dr^2 + r^2 d\phi^2$ is a catenoid of revolution

The equation of geodesics are given by

$\phi = \pm c \int \frac{\sqrt{1 + f'^2}}{r\sqrt{r^2 - c^2}} dr$ with $f(r) = b_0 \cosh^{-1}(\frac{r}{b_0})$

And we get $\phi = \pm c \int \frac{1}{\sqrt{r^2 - b_0^2} \sqrt{r^2 - c^2}} dr$

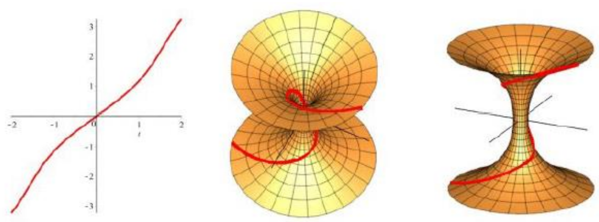

Fig. 6.4: Geodesics on Catenoid.