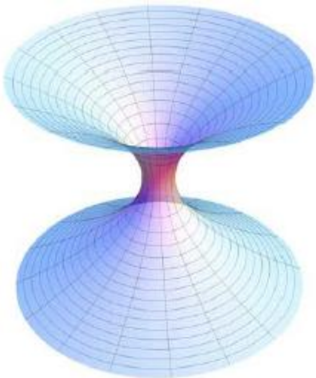
§ Morris-Thorne Wormhole 1988

$$ds^2 = -e^{2\phi}dt^2 + (1-\frac{b}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$\phi(r)$ ：redshift function。Change in frequency of electromagnetic radiation in gravitation field。

b(r)：shape function



properties of the metric：

1. Spherically symmetric and static
2. Radial coordinate r such that circumference of circle centered around throat given by 2πr
3. R decreases from $+\infty$ to $b = b_0$ (minimum radius) at throat，then increases from b_0 to $+\infty$
4. At throat exists coordinate singularity where r component diverges
5. Proper radial distance/r runs from $-\infty$ to $+\infty$ and vice versa

$$G_{ij} = 8\pi G T_{ij}$$

Cartan's Structure equations

$$d\omega^i = \omega^j \wedge \omega^i_j \text{ ， } \omega^k_j = \sum_i \Gamma^k_{ij}\omega^i$$

$$\Omega^j_i = d\omega^j_i - \omega^k_i \wedge \omega^j_k$$

$$\omega^0 = -e^{\phi}dt, \omega^1 = (1-\frac{b}{r})^{-\frac{1}{2}}dr, \omega^2 = rd\theta, \omega^3 = r\sin\theta d\phi$$

$$d\omega^0 = -e^{\phi}\phi' dr \wedge dt = \phi' e^{\phi} dt \wedge dr$$

$$d\omega^1 = 0$$

$$d\omega^2 = dr \wedge d\theta$$

$$d\omega^3 = \sin\theta dr \wedge d\phi + r\cos\theta d\theta \wedge d\phi$$

$$dt = -\frac{1}{e^{\phi}}\omega^0 \text{ ， } dr = (1-\frac{b}{r})^{\frac{1}{2}}\omega^1 \text{ ， } d\theta = \frac{1}{r}\omega^2 \text{ ， } d\phi = \frac{1}{r\sin\theta}\omega^3$$

$$d\omega^0 = \omega^1 \wedge \omega^0_1 + \omega^2 \wedge \omega^0_2 + \omega^3 \wedge \omega^0_3 = -\phi' e^{\phi} dr \wedge dt$$

$$\Rightarrow \omega^0_1 = -\phi' e^{\phi}(1-\frac{b}{r})^{\frac{1}{2}}dt, \omega^1_0 = \phi' e^{\phi}(1-\frac{b}{r})^{\frac{1}{2}}dt$$

同理 $\omega^2_1 = (1-\frac{b}{r})^{\frac{1}{2}}d\theta$ ， $\omega^3_1 = (1-\frac{b}{r})^{\frac{1}{2}}\sin\theta d\phi, \omega^3_2 = \cos\theta d\phi$

例如 $d\omega^3 = \omega^0 \wedge \omega^3_0 + \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2 = \sin\theta dr \wedge d\phi + r\cos\theta dr \wedge d\phi$

$\sin\theta dr\wedge d\phi=(1-\frac{b}{r})^{-\frac{1}{2}}dr\wedge\omega_1^3$ ， $r\cos\theta d\theta\wedge d\phi=rd\theta\wedge\omega_2^3$

得 $\omega_1^3=(1-\frac{b}{r})^{\frac{1}{2}}\sin\theta d\phi,\omega_2^3=\cos\theta d\phi$

$$\omega_i^j=\begin{bmatrix} 0 & -\phi' e^{\phi}(1-\frac{b}{r})^{\frac{1}{2}}dt & 0 & 0 \\ \phi' e^{\phi}(1-\frac{b}{r})^{\frac{1}{2}}dt & 0 & -(1-\frac{b}{r})^{\frac{1}{2}}d\theta & -\sin\theta(1-\frac{b}{r})^{\frac{1}{2}}d\phi \\ 0 & (1-\frac{b}{r})^{\frac{1}{2}}d\theta & 0 & -\cos\theta d\phi \\ 0 & \sin\theta(1-\frac{b}{r})^{\frac{1}{2}}d\phi & \cos\theta d\phi & 0 \end{bmatrix}$$

$$\Omega_i^j=d\omega_i^j-\omega_i^k\wedge\omega_k^j=-\frac{1}{2}R_{mni}^j\omega^m\wedge\omega^n$$

例如 $\Omega_3^2=d\omega_3^2-(\omega_3^0\wedge\omega_0^2+\omega_3^1\wedge\omega_1^2+\omega_3^2\wedge\omega_2^3+\omega_3^3\wedge\omega_2^3)$

$$=d(-\cos\theta d\phi)-(-\sin\theta(1-\frac{b}{r})^{\frac{1}{2}}d\phi\wedge(1-\frac{b}{r})^{\frac{1}{2}}d\theta)=\frac{b}{r}\sin\theta d\theta\wedge d\phi$$

$$=\frac{b}{r}\sin\theta\times\frac{1}{r}\times\frac{1}{r\sin\theta}\omega^2\wedge\omega^3=\frac{b}{r^3}\omega^2\wedge\omega^3$$

$$R_{323}^2=\Omega_3^2(E_2,E_3)=\frac{b}{r^3}$$

$$R_{101}^0=(1-\frac{b}{r})(-\phi''-(\phi')^2)-\frac{\phi'(b'r-b)}{2r^2}$$

$$R_{212}^1=R_{121}^2=-R_{112}^2=-R_{221}^1=\frac{b'r-b}{2r^3}$$

$$R_{220}^0=R_{020}^2=-R_{202}^0=-R_{002}^2=\frac{\phi'(1-\frac{b}{r})}{r}$$

$$R_{131}^3=R_{313}^1=-R_{331}^1=-R_{113}^3=\frac{b'r-b}{2r^3}$$

$$R_{330}^0=R_{030}^3=-R_{303}^0=-R_{003}^3=\frac{\phi'(1-\frac{b}{r})}{r}$$

$$R_{232}^3=R_{323}^2=-R_{223}^3=-R_{332}^2=\frac{b}{r^3}$$

$$R_{iji}^j=R_{jij}^i=-R_{iij}^j=-R_{jji}^i$$

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{jilk}$$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

Ricci curvature tensor

$$R_{ij} = \sum_k R^k_{ikj}$$

$$R_{00} = R^0_{101} + R^2_{200} + R^3_{030} = (1-\frac{b}{r})(\phi'' + (\phi')^2) + \frac{\phi'(b'r-b)}{2r^2} + \frac{2\phi'(1-\frac{b}{r})}{r}$$

$$R_{11} = R^0_{101} + R^1_{111} + R^2_{121} + R^3_{131} = (1-\frac{b}{r})(-\phi'' - (\phi')^2) - \frac{\phi'(b'r-b)}{2r^2} + \frac{b'r-b}{r^3}$$

$$R_{22} = R^0_{202} + R^2_{121} + R^3_{232} = \frac{-\phi'(1-\frac{b}{r})}{r} + \frac{b'r-b}{2r^3} + \frac{b}{r^3} = R_{33}$$

Curvature scalar

$$R = -R_{00} + R_{11} + R_{22} + R_{33}$$

$$= 2(1-\frac{b}{r})(-\phi'' - (\phi')^2) - \frac{\phi'(b'r-b)}{r^2} - \frac{4\phi'(1-\frac{b}{r})}{r} + \frac{2b'r}{r^3}$$

Einstein Tensor $G_{\mu\nu} = R_{\varpi\nu} - \frac{1}{2} g_{\mu\nu} R$ ， $R_{\mu\nu}$ is Ricci tensor，R is scalar curvature

Then $G_{00} =$

$G_{11} =$

$G_{22} =$

$G_{33} =$

$$G_{tt} = R_{tt} - \frac{1}{2}Rg_{tt} = \frac{b'}{r^2}$$

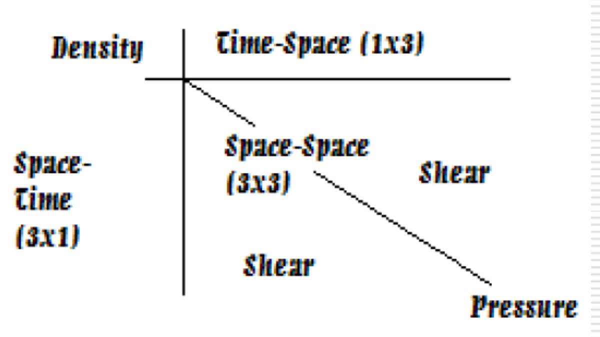
$$G_{rr} = R_{rr} - \frac{1}{2}Rg_{rr} = -\frac{b}{r^3} + \frac{2\Phi'}{r}\left(1-\frac{b}{r}\right)$$

$$G_{\theta\theta} = R_{\theta\theta} - \frac{1}{2}Rg_{\theta\theta} = \left(1-\frac{b}{r}\right)\left[\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right] = G_{\phi\phi}$$

$$G_{\phi\phi} = R_{\phi\phi} - \frac{1}{2}Rg_{\phi\phi} = \left(1-\frac{b}{r}\right)\left[\Phi'' - \frac{b'r-b}{2r(r-b)}\Phi' + (\Phi')^2 + \frac{\Phi'}{r} - \frac{b'r-b}{2r^2(r-b)}\right] = G_{\theta\theta}$$

Stress-Energy Tensor

$$T_{\alpha\beta} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & \tau & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$



Equation of State

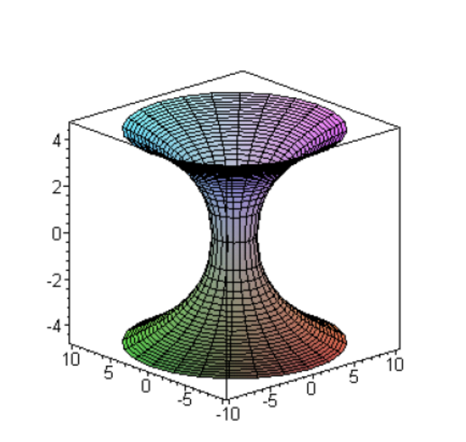
Energy density $\rho = \frac{b'}{8\pi r^2}$

Tension $\tau = \frac{1}{8\pi r^2}\left(\frac{b}{r} - 2(r-b)\phi'\right)$

Pressure(stress) $p = \frac{r}{2}\left|(\rho-\tau)\phi' - \tau'\right| - \tau$

Wormhole Embedding Diagram

$$z(r) = \pm b_0 \ln\left(\frac{r}{b_0} + \sqrt{\left(\frac{r}{b_0}\right)^2 - 1}\right),\ b_0 = 2$$



- Static (t=constant "slice")
- Assume $\theta=\pi/2$ (equatorial "slice")
- Only r,Φ variable

$$ds^2 = \left(1 - \frac{b}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

$$ds^2 = dz^2 + dr^2 + r^2 d\phi^2$$

Boundary conditions-Shape

Boundary conditions-No Horizon

Other boundary conditions

Geodesics

Variational Principle

Geodesics equations

Christoffel symbols

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{d\Phi}{dr} \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^r_{\phi\phi} = -(r-b)\sin^2\theta$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \qquad \Gamma^r_{\theta\theta} = b - r$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{\cos\theta}{\sin\theta} \qquad \Gamma^r_{tt} = \left(1 - \frac{b}{r}\right)\Phi' e^{2\Phi} \qquad \Gamma^r_{rr} = \frac{b'r - b}{2r(r-b)}$$