§ 6.6 Cosmology (FRW model)

Cartan：

$$d\omega^i=\sum_j \omega^j\wedge\omega_j^i \text{ ， } d\omega_i^j=\Omega_i^j+\sum_k \omega_i^k\wedge\omega_k^j$$

Take an isotropic 3-dim Riemannian manifold (Σ, h) as our model of space。

$h=a^2(\frac{1}{1-kr^2}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)$ ，where a>0 is the “radius” of the space and k=-1，0，1 according to whether the curvature is negative，zero or positive。

We take our model of the universe to be (M, g)，where $M=R\times\Sigma$ and

$$g=-dt^2+a^2(t)(\frac{1}{1-kr^2}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2)$$

These are the so-called Friedmann-Lemaitre-Robertson-Walker model of cosmology。

$$\omega^0=dt$$

$$\omega^r=a(t)(1-kr^2)^{-\frac{1}{2}}dr$$

$$\omega^\theta=a(t)rd\theta$$

$\omega^\varphi=a(t)r\sin\theta d\varphi$ ，here $\{\omega^0,\omega^r,\omega^\theta,\omega^\varphi\}$ is an orthonormal coframe。

$$\omega_r^0=\omega_0^r=\dot{a}\left(1-kr^2\right)^{-\frac{1}{2}}dr;$$
$$\omega_\theta^0=\omega_0^\theta=\dot{a}rd\theta;$$
$$\omega_\varphi^0=\omega_0^\varphi=\dot{a}r\sin\theta d\varphi;$$
$$\omega_r^\theta=-\omega_\theta^r=\left(1-kr^2\right)^{\frac{1}{2}}d\theta;$$
$$\omega_r^\varphi=-\omega_\varphi^r=\left(1-kr^2\right)^{\frac{1}{2}}\sin\theta d\varphi;$$
$$\omega_\theta^\varphi=-\omega_\varphi^\theta=\cos\theta d\varphi.$$

例如

$$d\omega^\varphi=\dot{a}(t)r\sin\theta dt\wedge d\varphi+a(t)\sin\theta dr\wedge d\varphi+a(t)r\cos\theta d\theta\wedge d\varphi$$

$=\omega^0\wedge\omega_0^\varphi+\omega^r\wedge\omega_r^\varphi+\omega^\theta\wedge\omega_\theta^\varphi$ 所以

$\omega_0^\varphi=\omega_\varphi^0=\dot{a}(t)r\sin\theta d\varphi$ ，$\omega_r^\varphi=-\omega_\varphi^r=(1-kr^2)^{-\frac{1}{2}}\sin\theta d\varphi$ ，$\omega_\theta^\varphi=-\omega_\varphi^\theta=\cos\theta d\varphi$

$$\Omega^0_r = \Omega^r_0 = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^r;$$
$$\Omega^0_\theta = \Omega^\theta_0 = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^\theta;$$
$$\Omega^0_\varphi = \Omega^\varphi_0 = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^\varphi;$$
$$\Omega^\theta_r = -\Omega^r_\theta = \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)\omega^\theta \wedge \omega^r;$$
$$\Omega^\varphi_r = -\Omega^r_\varphi = \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)\omega^\varphi \wedge \omega^r;$$
$$\Omega^\varphi_\theta = -\Omega^\theta_\varphi = \left(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}\right)\omega^\varphi \wedge \omega^\theta.$$

例如

$$\Omega^0_2 = \Omega^2_0 = d\omega^0_2 - (\omega^0_2 \wedge \omega^0_0 + \omega^1_2 \wedge \omega^0_1 + \omega^2_2 \wedge \omega^0_2 + \omega^3_2 \wedge \omega^0_3)$$

$$= \ddot{a}\, r dt \wedge d\theta = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^2$$

$$R^j_{iji} = \Omega^j_i(E_i, E_j), R_{ij} = \sum_k R^k_{kij}$$

$$\Omega^0_1 = \frac{\ddot{a}}{a}\omega^0 \wedge \omega^1 \text{ ， } R^0_{101} = \Omega^0_1(E_1, E_0) = -\frac{\ddot{a}}{a}$$

$$R^2_{121} = -(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2}) \text{ ， } R^3_{131} = -(\frac{k}{a^2} + \frac{\dot{a}^2}{a^2})$$

$$R_{11} = R^0_{011} + R^2_{211} + R^3_{311} = -(R^0_{101} + R^2_{121} + R^3_{131}) = \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}$$

同理 $R_{22} = R_{33} = R_{11}$

The nonvanishing components of Ric on this frame turn out to be

$$R_{00} = -\frac{3\ddot{a}}{a};$$
$$R_{rr} = R_{\theta\theta} = R_{\varphi\varphi} = \frac{\ddot{a}}{a} + \frac{2\dot{a}^2}{a^2} + \frac{2k}{a^2}.$$

At very large scales, galaxies and clusters of galaxies are expected to behave as particles of a pressureless fluid, which we take to be our matter model. By isotropy, the average spatial motion of the galaxies must vanish, and hence their unit velocity vector field must be $\frac{\partial}{\partial t}$ (corresponding to the 1-form $-dt$). Therefore the Einstein field equation is

$$Ric = 4\pi\rho(2dt \otimes dt + g),$$

which is equivalent to the ODE system

$$\begin{cases} -\dfrac{3\ddot{a}}{a} = 4\pi\rho \\ \\ \dfrac{\ddot{a}}{a} + \dfrac{2\dot{a}^2}{a^2} + \dfrac{2k}{a^2} = 4\pi\rho \end{cases} \Leftrightarrow \begin{cases} \ddot{a} + \dfrac{\dot{a}^2}{2a} + \dfrac{k}{2a} = 0 \\ \\ \rho = -\dfrac{3\ddot{a}}{4\pi a} \end{cases}.$$

後面還有一點解釋...