An Introduction to Riemannian Geometry CH 6.5 p.272

Spacetime and Geometry CH5.1 p.193

By Birkhoff Theorem：

In GR，the unique spherically symmertic vacuum solution is the Schwarzschild metric。

$$ds^2 = -(1-\frac{2GM}{r})dt^2 + (1-\frac{2GM}{r})^{-1}dr^2 + r^2 d\Omega^2$$

Where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$

Constant M is the mass of the gravitating object。

Since we are interested in the solution outside a sphrical body，we care about Einstein equation in vacuum， $R_{\mu\nu} = 0$

§ The Schwarzschild Solution

$$g = -A^2(r)dt^2 + B^2(r)dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2 d\varphi^2$$

For unknown positive smooth function A，B： $R \to R$

$$\omega^0 = A(r)dt, \omega^1 = B(r)dr, \omega^2 = rd\theta, \omega^3 = r\sin\theta d\varphi$$

$\{\omega^0, \omega^1, \omega^2, \omega^3\}$ is an orthonormal coframe。

$$d\omega^i = \sum_{j=0}^{3} \omega^j \wedge \omega^i_j$$

$\omega^0_0 = \omega^i_i = 0, \omega^0_i = \omega^i_0, \omega^i_j = -\omega^j_i$ on an orthonormal frame。

$$d\omega^0 = A'(r)dr \wedge dt = \frac{A'}{B}\omega^1 \wedge dt$$

$$= \omega^0 \wedge \omega^0_0 + \omega^1 \wedge \omega^0_1 + \omega^2 \wedge \omega^0_2 + \omega^3 \wedge \omega^0_3$$

$$\therefore \omega^0_1 = \frac{A'}{B}dt \text{ ， } \omega^0_2 = \omega^0_3 = 0$$

$$d\omega^1 = 0$$

$$d\omega^2 = dr \wedge d\theta = \frac{1}{B}\omega^1 \wedge d\theta$$

$$d\omega^3 = \sin\theta dr \wedge d\varphi + r\cos\theta d\theta \wedge d\varphi = \frac{\sin\theta}{B} w^1 \wedge d\varphi + \cos\theta\omega^2 \wedge d\varphi$$

$$= \omega^0 \wedge \omega^3_0 + \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2 + \omega^3 \wedge$$

$$\therefore \omega^3_1 = \frac{\sin\theta}{B} d\varphi$$

Yield the nonvanishing connection forms：

$$\omega_1^0 = \omega_0^1 = \frac{A'}{B}dt \ ; \ \omega_1^2 = -\omega_2^1 = \frac{1}{B}d\theta \ ; \ \omega_1^3 = -\omega_3^1 = \frac{\sin\theta}{B}d\varphi$$

$$\omega_2^3 = -\omega_3^2 = \cos\theta d\varphi$$

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j = \sum_{m<n} \mathrm{R}_{mni}^j \omega^m \wedge \omega^n$$

Then the curvature forms：

$$\Omega_r^0 = \Omega_0^r = \frac{A''B - A'B'}{AB^3}\omega^r \wedge \omega^0;$$
$$\Omega_\theta^0 = \Omega_0^\theta = \frac{A'}{rAB^2}\omega^\theta \wedge \omega^0;$$
$$\Omega_\varphi^0 = \Omega_0^\varphi = \frac{A'}{rAB^2}\omega^\varphi \wedge \omega^0;$$
$$\Omega_r^\theta = -\Omega_\theta^r = \frac{B'}{rB^3}\omega^\theta \wedge \omega^r;$$
$$\Omega_r^\varphi = -\Omega_\varphi^r = \frac{B'}{rB^3}\omega^\varphi \wedge \omega^r;$$

$$\Omega_\theta^\varphi = -\Omega_\varphi^\theta = \frac{B^2-1}{r^2B^2}\omega^\varphi \wedge \omega^\theta.$$

例如 $\Omega_r^\theta = \Omega_1^2 = d\omega_1^2 - \{\omega_1^0 \wedge \omega_0^2 + \omega_1^1 \wedge \omega_1^2 + \omega_1^2 \wedge \omega_2^2 + \omega_1^3 \wedge \omega_3^2\}$

$$= \frac{-B'}{B^2}dr \wedge d\theta = \frac{-B}{rB^3}\omega^1 \wedge \omega^2 \quad \text{as} \quad \omega^1 = \mathrm{B}dr, \omega^2 = rd\theta$$

$$\Omega_\theta^\varphi = \Omega_2^3 = d\omega_2^3 - (\omega_2^0 \wedge \omega_0^3 + \omega_2^1 \wedge \omega_1^3 + \omega_2^2 \wedge \omega_2^3 + \omega_2^3 \wedge \omega_3^3)$$

$$= d(\cos\theta d\varphi) - (-\frac{1}{B}d\theta \wedge \frac{\sin\theta}{B}d\varphi) = \frac{-1}{r^2}\omega^2 \wedge \omega^3 + \frac{1}{r^2B^2}\omega^2 \wedge \omega^3 = \frac{-B^2+1}{r^2B^2}\omega^2 \wedge \omega^3$$

Ricci tensor

$$R_{iji}^j = \Omega_i^j(E_i, E_j) \ R_{ij} = \sum_k R_{kij}^k$$

$$R_{00} = \mathrm{R}_{000}^0 + R_{100}^1 + R_{200}^2 + R_{300}^3 = -(R_{010}^1 + R_{020}^2 + R_{030}^3)$$

$$= \frac{A''B - A'B'}{AB^3} + \frac{2A'}{rAB^2}$$

Where

$$R_{010}^1 = \Omega_0^1(E_0, E_1) = -\frac{A''B - A'B'}{AB^3} \ , \ R_{020}^2 = \Omega_0^\theta(E_0, E_\theta) = -\frac{A'}{rAB^2}$$

$$R^3_{030} = \Omega^{\varphi}_0(E_0, E_\varphi) = \frac{-A'}{rAB^2}$$

$$R_{11} = R^0_{011} + R^1_{111} + R^2_{211} + R^3_{311} = -(R^0_{101} + R^2_{121} + R^3_{131}) = \frac{-A''B + A'B'}{AB^3} + \frac{2B'}{rB^3}$$

Where

$$R^0_{101} = \Omega^0_1(E^1, E^0) = \frac{-A''B + A'B'}{AB^3} \text{ , } R^2_{121} = \Omega^2_1(E_1, E_2) = \frac{-B'}{rB^3} \text{ ,}$$

$$R^3_{131} = \Omega^3_1(E_1, E_3) = \frac{-B'}{rB^3}$$

$$R_{22} = R^0_{022} + R^1_{122} + R^2_{222} + R^3_{322} = -R^0_{202} - R^1_{212} - R^3_{232}$$

$$= -\frac{A'}{rAB^2} + \frac{B'}{rB^3} + \frac{B^2-}{r^2}$$

Where

$$R^0_{202} = \Omega^0_\theta(E_\theta, E_0) = \frac{A'}{rAB^2} \text{ , } R^1_{212} = \Omega^r_\theta(E_\theta, E_r) = \frac{-B'}{rB^3} \text{ , } R^3_{232} = \Omega^{\varphi}_\theta(E_\theta, E_\varphi) = -\frac{B^2-1}{r^2B^2}$$

$$R_{33} = R^0_{033} + R^1_{133} + R^2_{233} + R^3_{333} = R_{22}$$

$$R_{00} = \frac{A''B - A'B'}{AB^3} + \frac{2A'}{rAB^2};$$

$$R_{rr} = -\frac{A''B - A'B'}{AB^3} + \frac{2B'}{rB^3};$$

$$R_{\theta\theta} = R_{\varphi\varphi} = -\frac{A'}{rAB^2} + \frac{B'}{rB^3} + \frac{B^2-1}{r^2B^2}.$$

Thus the vacuum Einstein field equation $Ric = 0$ is equivalent to the ODE system

$$\begin{cases} \dfrac{A''}{A} - \dfrac{A'B'}{AB} + \dfrac{2A'}{rA} = 0 \\[2ex] \dfrac{A''}{A} - \dfrac{A'B'}{AB} - \dfrac{2B'}{rB} = 0 \\[2ex] \dfrac{A'}{A} - \dfrac{B'}{B} - \dfrac{B^2-1}{r} = 0 \end{cases} \Leftrightarrow \begin{cases} \dfrac{A'}{A} + \dfrac{B'}{B} = 0 \\[2ex] \left(\dfrac{A'}{A}\right)' + 2\left(\dfrac{A'}{A}\right)^2 + \dfrac{2A'}{rA} = 0 \\[2ex] \dfrac{2B'}{B} + \dfrac{B^2-1}{r} = 0 \end{cases} .$$

The last equation can be immediately solved to yield $B=(1-\frac{2m}{r})^{-\frac{1}{2}}$，where $m\in R$ is an integration constant。

The first equation implies that $A=\frac{\alpha}{B}$ for some constant $\alpha>0$。

By rescaling the time coordinate t we can assume that $\alpha=1$

Finally，it is easy to check that the second ODE is identically satisfied。

Therefore there exists a one-parameter family of solution of vacuum Einstein field equation of the form we sought，given by

$$g=-(1-\frac{2m}{r})dt^2+(1-\frac{2m}{r})^{-1}dr^2+r^2d\theta^2+r^2\sin^2\theta d\varphi^2$$

§ 5.6 Schwarzschild black hole [Spacetime and Geometry] p.218