§ Cartan Computations

Let $\mathrm{E}_1,\ldots E_n$ be an orthonormal frame field，$\omega^1,\ldots,\omega^n$，its dual forms is the orthonormal coframe field with $\omega^i(E_j)=\delta^i_j$

Definition：

The connection forms of a frame field $E_1,\ldots,E_n$ are the one-forms ω^i_j such that for all tangent vectors v，$\omega^i_j(v)=\omega^i(\nabla_v E_j)$ for all i，j

Hence by the duality formula，$\nabla_v E_j=\sum \omega^m_j(v)E_m$

$\omega^k_j := \sum_i \Gamma^k_{ij}\omega^i$ ，$R_{vw}(E_j)=\sum_i \Omega^i_j(v,w)E_i$ ，for tangent vectors v，w

Where Ω^i_j is a two-form called the curvature forms of the frame field。

1. $d\omega^i=\sum_j \omega^j\wedge\omega^i_j$
2. $\Omega^j_i=d\omega^j_i-\sum_k \omega^k_i\wedge\omega^j_k$
3. $R^i_{jkl}=\Omega^i_j(E_k,E_l)$ curvature tensor

The sectional curvatures of the frame field 2-planes are K $K(\Pi_{ij})=K(E_i,E_j)=\varepsilon_j R^i_{jij}$

Where $\varepsilon_i=\langle \mathrm{E}_i,E_i\rangle$

$\Omega^i_j=-\varepsilon_i\varepsilon_j\Omega^j_i$ 。In particular，$\Omega^i_i=0$

Remarks：

Let $\{E_i\}$ be an orthonormal fram field onM

1. If M is a Riemannian manifold，the signs ε_i are all $+1$，(ω^i_j) and (Ω^i_j) are skew-symmetric。
2. If M is a Lorentz manifold，then $\omega^0_i=\omega^i_0$，but $\omega^i_j=-\omega^j_i$ for i，j>0，and similarly for Ω^i_j

Let u,v be orthogonal coordinates in a semi-Riemannian surface S。

$\langle \partial_u, \partial_u \rangle = E = \varepsilon_1 e^2, \langle \partial_v, \partial_v \rangle = G = \varepsilon_2 g^2$ ，where $\varepsilon_1, \varepsilon_2$ are ± 1，e>0，g>0。

Then the Gaussian curvature of S ，$K_S = \dfrac{-1}{eg}\left[\varepsilon_1\left(\dfrac{e_v}{g}\right)_v + \varepsilon_2\left(\dfrac{g_u}{e}\right)_u\right]$

Prove

Define $E_1 = \dfrac{\partial_u}{e}, E_2 = \dfrac{\partial_v}{g}$ ，an orthonormal fram field

Then the Gaussian curvature $K_S = \langle \Omega_2^1(E_1, E_2)E_1, E_1 \rangle = \varepsilon_1 \Omega_2^1(E_1, E_2)$

The dual coframe is $\omega^1 = edu, \omega^2 = gdv$

$$d\omega^1 = e_v dv \wedge du = -\frac{e_v}{g} du \wedge \omega^2 = \omega^2 \wedge \omega_2^1$$

$$d\omega^2 = g_u du \wedge dv = -\frac{g_u}{e} dv \wedge \omega^1 = \omega^1 \wedge \omega_2^1$$

$$\omega_2^1 = \frac{e_v}{g} du - \varepsilon_1 \varepsilon_2 \frac{g_u}{e} dv \quad (\omega_2^1 = -\varepsilon_1 \varepsilon_2 \omega_1^2)$$

$$\Omega_1^2 = d\omega_1^2$$

$$d\omega_2^1 = (\frac{e_v}{g})_v dv \wedge du - \varepsilon_1 \varepsilon_2 (\frac{g_u}{e})_u du \wedge dv$$

$$= -[(\frac{e_v}{g})_v + \varepsilon_1 \varepsilon_2 (\frac{g_u}{e})_u] du \wedge dv$$

$$K_S = \varepsilon_1 \Omega_2^1(E_1, E_2) = \varepsilon_1 d\omega_2^1(E_1, E_2) = \varepsilon_1 d\omega_2^1(\frac{\partial_u}{e}, \frac{\partial v}{g})$$

$$= -\varepsilon_1 [(\frac{e_v}{g})_v + \varepsilon_1 \varepsilon_2 (\frac{g_u}{e})_u] / (eg)$$

In classical differential geometry $K = \dfrac{eg - f^2}{EG - F^2}$

Where $E = X_u \cdot X_u, G = X_v \cdot X_v, F = X_u \cdot X_v$

$e = X_{uu} \cdot N, g = X_{vv} \cdot N, f = X_{uv} \cdot N$

在測地坐標系中，$ds^2 = du^2 + g^2 dv^2$ ，$X_u \cdot X_u = 1$ ，$X_v \cdot X_v = g^2$

從 $\dfrac{\partial}{\partial u} X_{uv} = \dfrac{\partial}{\partial v} X_{uu}$ 出發，兩邊對 X_v 做內積，可以推出 $K = -\dfrac{g_{uu}}{g}$

$$E_1 = \frac{\partial}{\partial u}, E_2 = \frac{1}{g}\frac{\partial}{\partial v} \text{ ，} \omega^1 = du, \omega^2 = gdv$$

$$d\omega^1 = 0$$

$$d\omega^2 = g_u du \wedge dv = \omega^1 \wedge \omega_1^2 \text{ ，} \omega_1^2 = g_u dv$$

$$\Omega_1^2 = d\omega_1^2 = g_{uu} du \wedge dv$$

$$K = \Omega_2^1(E_1, E_2) = -\Omega_1^2(E_1, E_2) = -\frac{g_{uu}}{g}$$