Prove that $d\omega(X,Y)=\frac{1}{2}\{X\langle\omega,Y\rangle-Y\langle\omega,X\rangle-\langle\omega,[X,Y]\rangle\}$

c.f. RG2101differentialforms

$X\in T_pM$ is a contravariant vector

$X=X^i\frac{\partial}{\partial x^i}$，then $Xf=X^i\frac{\partial f}{\partial x^i}$。

For examle $X=y\partial_x-x\partial_y$，$f(x,y)=x^2y+y^2$。

Then $Xf=y(2xy)-x(x^2+2y)=2xy^2-x^3-2xy$

If $\{e_j\}$ is a basis，$e_j=\Phi^i_j\frac{\partial}{\partial x^i}$，thwn $X=X^je_j$

T^*_pM is the dual space with basis $\{e^i\}$，$e^i(e_j)=\delta^i_j$

$\omega=\omega_ie^i$ with $e^i=\Phi^i_kdx^k$

Then $\{e_i\}$ and $\{e^i\}$ are called canonical (典範)basis

For a function f，$df(X)=\langle df,X\rangle=Xf=X^i\frac{\partial f}{\partial x^i}$

$\omega(X):=\langle\omega,X\rangle$

Example

1. $f,g:R^2\to R$

   $df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy, dg=\frac{\partial g}{\partial x}dx+\frac{\partial g}{\partial y}dy$

   $df\wedge dg=\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{vmatrix}dx\wedge dy$，the former is the Jacobian，and the later is the area form。

2. $\omega=P(dy\wedge dz)+Q(dz\wedge dx)+R(dx\wedge dy)$

   $X=\xi^1\partial_x+\xi^2\partial_y+\xi^3\partial_z$，$Y=\zeta^1\partial_x+\zeta^2\partial_y+\zeta^3\partial_z$

   Then $\omega\bullet(X\wedge Y)=\left\|\begin{bmatrix}P & Q & R\\ \xi^1 & \xi^2 & \xi^3\\ \zeta^1 & \zeta^2 & \zeta^3\end{bmatrix}\right\|$

If ω is a p-form，η is a q-form，then $d(\omega\wedge\eta)=d\omega\wedge\eta+(-1)^p\omega\wedge d\eta$

§ Lie bracket and Lie derivative

$[X,Y]=XY-YX$ or $[X,Y]^j=X^kY^j_{,k}-Y^kX^j_{,k}$ ...(*)

Jacobi identity $[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$

$L_XY=[X,Y]$

1. $L_Xf=Xf=df(X)$
2. $L_X(S\otimes T)=L_XS\otimes T+S\otimes L_XT$
3. $L_X\langle\omega,Y\rangle=\langle L_X\omega,Y\rangle+\langle\omega,L_XY\rangle$

i.e. $L_X(\omega Y)=(L_X\omega)Y+\omega(L_XY)$ or $X^k(\omega_jY^j)_{,k}=(L_X\omega)_jY^j+\omega_j(L_XY)^j$

其中，由(*) $(L_X\omega)_jY^j=X^k(\omega_{j,k}Y^j+\omega_jY^j_{,k})-\omega_j(X^kY^j_{,k}-Y^kX^j_{,k})$

$=(X^k\omega_{j,k}-\omega_kX^k_{,j})Y^j$，then $(L_X\omega)_j=\omega_{j,k}X^k+\omega_kX^k_{,j}$

$\langle L_X\omega,Y\rangle-Y\langle\omega,X\rangle=(\omega_{j,k}X^k+\omega_kX^k_{,j})Y^j-Y^j(\omega_{k,j}X^k+\omega_kX^k_{,j})$

$=(\omega_{j,k}-\omega_{k,j})X^kY^j=2d\omega(X,Y)$

Then $d\omega(X,Y)=\dfrac{1}{2}\{L_X\langle\omega,Y\rangle-Y\langle\omega,X\rangle-\langle\omega,[X,Y]\rangle\}$

Let $\omega=h_1dx^1+\ldots+h_ndx^n$，$X=\xi^1\partial_{x^1}+\ldots+\xi^n\partial_{x^n}$

Prove that $L_X\omega=\sum_j(Xh_j)dx^i+\sum_kh_kd\xi^k$