§ Ricci flow soliton

Ricci flow $\frac{\partial g(t)}{\partial t} = -2Ric(g(t))$

The flow evolves the metric in the direction of decreasing Ricci curvature，analogous to how heat flow smooths out temperature distributions。

Ricci soliton

A metric g is a Ricci soliton if it satisfies the equation：

$$Ric(g) + \frac{1}{2} L_X g = \lambda g$$

If the vector field X is the gradient of a function f，then the soliton is called a gradient Ricci soliton，and the equation becomes：

$Ric(g) + \nabla\nabla f = \lambda g$ where $\nabla\nabla f$ is the Hessian of f。

Interpretation

1. Ricci solitons are self-similar solutions to the Ricci flow，meaning they evolve by scaling and diffeomorphisms rather than changing their shape。
2. They play a crucial role in understanding the long-time behavior of the Ricci flow and the formation of singularities。
3. Examples of Ricci solitons include Einstein metrics (where $\mathrm{Ric}(g)=\lambda g$) and certain homogeneous spaces.

Ricci solitons are important in geometric analysis because：

1. They provide models for singularities that can form under the Ricci flow。
2. They are critical points of certain functionals related to the Ricci flow。
3. They help classify manifolds and understand their geometric structures。

Examples

1. Gaussian solitons (flat Euclidean space)

Consider the standard flat metric on $\mathbb{R}^n$:

$$g = \delta_{ij}\, dx^i \otimes dx^j$$

In this case:

- The **Ricci curvature** $\mathrm{Ric}(g) = 0$ because flat space has zero curvature.
- Choose the vector field $X = \frac{1}{2}\mathbf{x} = \frac{1}{2}\sum_i x^i \frac{\partial}{\partial x^i}$.
- The **Lie derivative** $\mathcal{L}_X g = g$, meaning the metric is scaling under this flow.

This satisfies the **shrinking soliton** equation:

$$\mathrm{Ric}(g) + \mathcal{L}_X g = \frac{1}{2} g$$

2. Cigar soliton(2D steady soliton)

In two dimensions, the **cigar soliton** is another important example:

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

- **Ricci curvature**: Decays away from the origin.
- **Vector field**: Points radially outward.
- **Type**: Steady soliton ($\lambda = 0$).

It looks like a "cigar" shape and represents a steady-state solution to the Ricci flow in 2D.

3. Sphere as a shrinking soliton

The standard round sphere S^n with its usual metric is an example of a **shrinking Ricci soliton**.

- **Ricci curvature**: $\mathrm{Ric}(g) = (n-1)g$
- **Vector field**: Zero (trivial diffeomorphism contribution).
- **Type**: Shrinking ($\lambda = n - 1$).

The sphere contracts to a point under Ricci flow.

§ Ricci flow solitons and singularity formation

There are three types of singularities that can develop in Ricci flow：

Perelman insights：Ricci solitons and surgery

1. Canonical form of singurarities
2. Ricci flow with surgery
3. Entropy and W-functional

§ Examples：singularities forming under Ricci flow

1. Ricci flow on a sphere S^n
2. Neckpinch singularity (collapsing cylider to two spheres)
3. Perelman surgery and the Poincare conjeture

Visual intuition

Imagine a sphere made of rubber：

1. **Ricci flow** behaves like heat diffusion，smoothing out irregularities。
2. **If the sphere is perfectly round**，it shrinks evenly like a balloon deflating。
3. **If it's elongated (dumbbell shape)**，the thin neck shrinks faster，leading to a **neckpinch singularity**。