Chapter 2

The Ricci Soliton Equation

In this chapter we familiarize ourselves with the Ricci soliton equation. In particular, we see how Ricci solitons are dynamically self-similar solutions to the Ricci flow and we consider special examples. We consider the special case of gradient Ricci solitons, which are the main objects of study in this book. By differentiating the Ricci soliton equation, we derive fundamental and useful identities. Regarding the qualitative study of Ricci solitons, we discuss the lower bound for the scalar curvature, completeness of the Ricci soliton vector field, and the uniqueness theorem for compact Ricci solitons.

A **Ricci soliton structure** is a quadruple $(\mathcal{M}^n, g, X, \lambda)$ consisting of a smooth manifold $\mathcal{M}^n$, a Riemannian metric g, a smooth vector field X, and a real constant λ, which together satisfy the equation

$$\operatorname{Ric} + \frac{1}{2}\mathcal{L}_X g = \frac{\lambda}{2} g \tag{2.1}$$

on $\mathcal{M}^n$, where Ric denotes the Ricci tensor of g and where $\mathcal{L}$ denotes the Lie derivative. We include the factor of one half in order to slightly simplify certain fundamental equations which follow.

Tracing (2.1), we have

$$R + \operatorname{div} X = \frac{n\lambda}{2}, \tag{2.2}$$

where R is the scalar curvature of g and $\operatorname{div} X = \operatorname{tr}(\nabla X) = \sum_{i=1}^n \nabla_i X^i$ denotes the divergence of X; cf. (1.43). Here, ∇ is the Riemannian covariant derivative.

Note that when we write ∇f, where f is a function, this could mean either (1) the covariant derivative, which is equal to the exterior derivative,

$\nabla f = df$, or (2) the gradient ∇f, which is the vector field metrically dual to the 1-form df. In local coordinates,

$$\nabla_i f := (df)_i = \frac{\partial f}{\partial x^i} \quad \text{and} \quad \nabla^i f := (\nabla f)^i = g^{ij}\nabla_j f.$$

The most important class of Ricci solitons, and the primary focus of this book, is those for which $X = \nabla f$ for some smooth function f on $\mathcal{M}^n$. For these so-called **gradient Ricci solitons**, equation (2.1) simplifies to

$$\mathrm{Ric} + \nabla^2 f = \frac{\lambda}{2} g, \tag{2.3}$$

since $\mathcal{L}_{\nabla f} g = 2\nabla^2 f$ (see (2.28) below if you have not seen this formula). Here, ∇^2 denotes the Hessian, i.e., the second covariant derivative. This acts on tensors, and when acting on a function f, $\nabla^2 f = \nabla df$. We will often use the abbreviation **GRS** for gradient Ricci soliton.

We will use the notation $(\mathcal{M}^n, g, f, \lambda)$ to denote a gradient Ricci soliton structure. When the **expansion constant** (or **scale**) λ is fixed and the **potential function** f is known or can be determined from the context at hand, we will often simply refer to the underlying manifold $(\mathcal{M}^n, g)$ as *the* Ricci soliton.[1]

2.1. Riemannian symmetries and notions of equivalence

The groups $\mathbb{R}_+$ of positive real numbers and $\mathrm{Diff}(\mathcal{M}^n)$ of diffeomorphisms act naturally by dilation via $\alpha \cdot g = \alpha g$ and pull back via $\phi \cdot g = \phi^* g$, respectively, on the space $\mathrm{Met}(\mathcal{M}^n)$ of Riemannian metrics on $\mathcal{M}^n$. Via the scaling and diffeomorphism invariances

$$\mathrm{Ric}(\alpha g) = \mathrm{Ric}(g), \quad \mathrm{Ric}(\phi^* g) = \phi^* \mathrm{Ric}(g) \tag{2.4}$$

of the Ricci tensor, they act on Ricci solitons $(\mathcal{M}^n, g, X, \lambda)$ as follows:

(1) (Metric scaling) If $\alpha \in \mathbb{R}_+$, then $(\mathcal{M}^n, \alpha g, \alpha^{-1} X, \alpha^{-1}\lambda)$ is a Ricci soliton.

(2) (Diffeomorphism invariance) If $\varphi : \mathcal{N}^n \to \mathcal{M}^n$ is a diffeomorphism, then $(\mathcal{N}^n, \varphi^* g, \varphi^* X, \lambda)$ is a Ricci soliton.

Observe also that if K is a Killing vector field, then $(\mathcal{M}^n, g, X + K, \lambda)$ is a Ricci soliton. We leave it as an exercise to check these properties (see Exercise 2.6). Only the *sign* of the expansion constant λ is of material significance, since, according to property (1), we can adjust the magnitude of a nonzero λ arbitrarily by multiplying g and X by appropriate positive

[1] In the case where $\mathcal{M}^n$ is closed, the function f is the same as the potential function defined by (1.22) since by tracing (2.3) we have that $\Delta f = \frac{n\lambda}{2} - R$ and $\frac{n\lambda}{2}$ must be equal to the average scalar curvature.

factors. We will see shortly that each Ricci soliton gives rise at least to a locally defined *self-similar* solution to the Ricci flow, with the scaling behavior determined by whether λ is positive, negative, or zero. This characteristic scaling behavior motivates the following terminology.

Definition 2.1 (Types of Ricci solitons). A Ricci soliton $(\mathcal{M}^n, g, X, \lambda)$ is said to be **shrinking** if $\lambda > 0$, **expanding** if $\lambda < 0$, and **steady** if $\lambda = 0$.

For brevity, we will often simply refer to such Ricci solitons as **shrinkers**, **expanders**, or **steadies**. When working within one of these classes of Ricci solitons, we will usually normalize the structure so that λ is 1, -1, or 0 and suppress further mention of it.[2] For example, the shrinking GRS equation is

$$\mathrm{Ric} + \nabla^2 f = \frac{1}{2} g. \tag{2.5}$$

In §2.2 we will see, via the equivalent dynamical version of Ricci solitons, the reasons for the terminologies shrinking, expanding, and steady.

We will say that two Ricci soliton structures $(\mathcal{M}_i^n, g_i, X_i, \lambda_i)$, $i = 1, 2$, are **equivalent** if $\lambda_1 = \lambda_2$ and the underlying Riemannian manifolds $(\mathcal{M}_i^n, g_i)$ are isometric. An isometry $\phi : (\mathcal{M}_1^n, g_1) \to (\mathcal{M}_2^n, g_2)$ need not pull back X_2 to X_1, however, since

$$\mathrm{Ric}(g_1) - \frac{\lambda_1}{2} g_1 = \phi^* \left(\mathrm{Ric}(g_2) - \frac{\lambda_2}{2} g_2 \right), \tag{2.6}$$

and we have (see Exercise 2.3)

$$\mathcal{L}_{X_1} g_1 = \phi^* (\mathcal{L}_{X_2} g_2) = \mathcal{L}_{\phi^* X_2} \phi^* g_2 = \mathcal{L}_{\phi^* X_2} g_1,$$

so

$$\mathcal{L}_{(\phi^* X_2 - X_1)} g_1 = 0; \tag{2.7}$$

i.e., the difference $\phi^* X_2 - X_1$ will at least be a Killing vector field on $(\mathcal{M}_1^n, g_1)$. In particular, it is not difficult to see that $(\mathcal{M}^n, g, X_1, \lambda)$ and $(\mathcal{M}^n, g, X_2, \lambda)$ are equivalent if and only if $X_2 - X_1$ is a Killing vector field.

2.2. Ricci solitons and Ricci flow self-similarity

The scaling and diffeomorphism invariances of the Ricci tensor (2.4) manifest themselves in symmetries of the Ricci flow equation. If $g(t)$ is a solution to the Ricci flow on $\mathcal{M}^n \times [c, d]$, then, for any fixed $\alpha > 0$ and $\phi \in \mathrm{Diff}(\mathcal{M}^n)$,

$$\tilde{g}(t) := \alpha (\phi^* g)(t/\alpha)$$

is a solution on $\mathcal{M}^n \times [\alpha c, \alpha d]$. From a geometric perspective, these solutions are essentially the same: For each t, $g(t/\alpha)$ and $\tilde{g}(t)$ are isometric but for a

[2]Strictly speaking, no normalization is required if $\lambda = 0$.

homothetical constant. A solution to the Ricci flow which moves exclusively under these symmetries, that is, which has the form

$$g(t) = c(t)\,\phi_t^*\bar{g} \tag{2.8}$$

for some fixed metric $\bar{g}$ and positive smooth function $c(t)$ and smooth family of diffeomorphisms ϕ_t, is therefore essentially stationary from a geometric perspective. To wit, Ricci solitons are the fixed points of the Ricci flow in the space of metrics modulo scalings and diffeomorphisms. Such solutions are said to be **self-similar**.

The following proposition demonstrates that Ricci solitons and self-similar solutions are two sides of the same coin: A self-similar solution defines a Ricci soliton structure on each time-slice, and conversely a Ricci soliton structure gives rise to an (at least locally defined) self-similar solution.[3] The interplay between the two perspectives, one static and one dynamic, is fundamental to the analysis of Ricci solitons. The following is our first formulation; we reformulate it slightly later.

Proposition 2.2 (Canonical form, I). *Let $(\mathcal{M}^n, g_0)$ be a Riemannian manifold.*

(a) *Suppose that $g(t) = c(t)\phi_t^* g_0$ satisfies the Ricci flow on $\mathcal{M}^n \times (\alpha, \omega)$ for some positive smooth function $c : (\alpha, \omega) \to \mathbb{R}$ and smooth family of diffeomorphisms $\{\phi_t\}_{t\in(\alpha,\omega)}$. Then, for each $t \in (\alpha, \omega)$, there is a vector field $X(t)$ and a scalar $\lambda(t)$ such that $(\mathcal{M}^n, g(t), X(t), \lambda(t))$ satisfies the Ricci soliton equation* (2.1).

(b) *Suppose that $(\mathcal{M}^n, g_0, X, \lambda)$ satisfies the Ricci soliton equation* (2.1) *for some smooth vector field X and constant λ. Then, for each $x_0 \in \mathcal{M}^n$, there is a neighborhood U of x_0, an interval (α, ω) containing 0, a smooth family $\phi_t : U \to \mathcal{M}^n$ of injective local diffeomorphisms, and a smooth positive function $c : (\alpha, \omega) \to \mathbb{R}$ such that $g(t) = c(t)\phi_t^* g_0$ solves the Ricci flow on $U \times (\alpha, \omega)$ with $g(0) = g_0$.*

Proof. Suppose first that $g(t) = c(t)\phi_t^* g_0$ solves the Ricci flow on $\mathcal{M}^n \times (\alpha, \omega)$. Fix $a \in (\alpha, \omega)$. Differentiating $g(t)$ at a yields

$$\left.\frac{\partial}{\partial t}\right|_{t=a} g(t) = c'(a)\phi_a^* g_0 + c(a)\left.\frac{\partial}{\partial t}\right|_{t=a} \phi_t^* g_0.$$

Now,

$$\left.\frac{\partial}{\partial t}\right|_{t=a} \phi_t^* g_0 = \left.\frac{\partial}{\partial t}\right|_{t=0} (\phi_a^{-1} \circ \phi_{a+t})^* \phi_a^* g_0 = \mathcal{L}_{X(a)} \phi_a^* g_0,$$

[3] If g is complete, then one obtains a globally defined self-similar solution; see Theorem 2.27 below.

where $X(a)$ is the generator of the family $\phi_a^{-1} \circ \phi_{a+t}$, so, taking $\lambda(a) = -c'(a)/c(a)$ and using that $g(t)$ solves the Ricci flow, we obtain a solution $(\mathcal{M}^n, g(a), X(a), \lambda(a))$ to the Ricci soliton equation (2.1).

On the other hand, suppose that $(\mathcal{M}^n, g_0, X, \lambda)$ satisfies (2.1) and that $x_0 \in \mathcal{M}^n$. By the local existence theory for ODEs (see, for example, Theorem 9.12 of [**216**]), there are open neighborhoods U, V of x_0 with $U \subset V$, $\epsilon > 0$, and a smooth family of injective local diffeomorphisms $\psi_s : U \to V$, $s \in (-\epsilon, \epsilon)$, such that $\psi_0(x) = x$ and

$$\left.\frac{\partial}{\partial s}\right|_{s=a} \psi_s(x) = X(\psi_a(x))$$

on $U \times (-\epsilon, \epsilon)$.

When $\lambda \neq 0$, define $\omega = \min\{\epsilon, |\lambda|\}$ and $\alpha = -\omega$, and, for $t \in (\alpha, \omega)$, let

$$c(t) = 1 - \lambda t, \quad \phi_t = \psi_{s(t)},$$

where

$$s(t) = -\frac{1}{\lambda} \ln(1 - \lambda t).$$

Then $g(t) = c(t)\phi_t^* g_0$ satisfies $g(0) = g_0$ and

$$\begin{aligned}\frac{\partial g}{\partial t} &= c'(t)\psi_{s(t)}^* g_0 + c(t)s'(t)\psi_{s(t)}^* \mathcal{L}_X g_0 \\ &= -\lambda \phi_t^* g_0 + \phi_t^*(-2\mathrm{Ric}(g_0) + \lambda g_0) \\ &= -2\mathrm{Ric}(g(t))\end{aligned}$$

on $U \times (\alpha, \omega)$.

When $\lambda = 0$,

$$\frac{\partial}{\partial t}\psi_t^* g_0 = \psi_t^* \mathcal{L}_X g_0 = -2\psi_t^* \mathrm{Ric}(g_0) = -2\mathrm{Ric}(g(t))$$

on $U \times (-\epsilon, \epsilon)$ so (b) is verified in this case with $c(t) = 1$ and $\phi_t = \psi_t$. □

The interval of existence of the solution in the second half of the above proposition is constrained by the maximum domain of definition of the 1-parameter family of diffeomorphisms generated by the vector field X. However, as we will see in §2.8 below, the vector field X will in most cases of interest generate a flow for all $t \in \mathbb{R}$ (i.e., X is a complete vector field), and in these settings the correspondence between self-similar solutions and Ricci solitons is symmetric.

When the vector field X generates a global flow, the interval of definition for the self-similar solution will be at least as large as that permitted by the Ricci soliton type, namely, $(-\infty, \lambda^{-1})$ for shrinkers, $(-\infty, \infty)$ for steadies, and $(-\lambda^{-1}, \infty)$ for expanders. The lifetime of a self-similar solution may extend beyond these intervals. This phenomenon occurs, for example, in the shrinking and expanding self-similar solutions arising from the Gaussian soliton; see (2.9) immediately below.

2.3. Special and explicitly defined Ricci solitons

In this section we consider some important examples and special classes of Ricci solitons.

2.3.1. The Gaussian soliton.

For $\lambda \in \mathbb{R}$, the structure $(\mathbb{R}^n, g_{\mathrm{Euc}}, f_{\mathrm{Gau}}, \lambda)$, where

$$g_{\mathrm{Euc}} = \sum_{i=1}^{n} dx^i \otimes dx^i \quad \text{and} \quad f_{\mathrm{Gau}}(x) = \frac{\lambda}{4} |x|^2, \tag{2.9}$$

is called the **Gaussian soliton**. Thus, Euclidean space can be regarded as a Ricci soliton of shrinking, expanding, or steady type. Observe that the choice of potential function $f = f_{\mathrm{Gau}}$ is not unique: Any function of the form $f(x) = \frac{\lambda}{4} |x|^2 + \langle a, x \rangle + b$, where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, yields an equivalent Ricci soliton structure.

The self-similar solution to the Ricci flow associated to the Gaussian soliton is static for any choice of λ. It is instructive to carry out the construction in Proposition 2.2 for this simple case explicitly. Integrating the vector field

$$\nabla f = \frac{\lambda x^i}{2} \frac{\partial}{\partial x^i} \tag{2.10}$$

produces the 1-parameter family of diffeomorphisms $\tilde{\phi}_t(x) = \mathrm{e}^{\frac{\lambda t}{2}} x$. Following Proposition 2.2 and taking $\phi_t = \tilde{\phi}_{-\lambda^{-1} \ln(1-\lambda t)}$ when $\lambda \neq 0$ and $\phi_t = \tilde{\phi}_t$ when $\lambda = 0$, we find that

$$\phi_t(x) = (1 - \lambda t)^{-1/2} x \tag{2.11}$$

and hence that the associated solution $g(t)$ is

$$g(t) = (1 - \lambda t)\, \phi_t^* g_{\mathrm{Euc}} = g_{\mathrm{Euc}}. \tag{2.12}$$

When $\lambda \neq 0$, the family of diffeomorphisms ϕ_t — and by extension, the solution provided by Proposition 2.2 — is defined only for $t \in (-\infty, \lambda^{-1})$ or $t \in (\lambda^{-1}, \infty)$ depending on whether λ is positive or negative. However, the solution $g(t)$ is well-defined by the rightmost expression for all $t \in (-\infty, \infty)$.

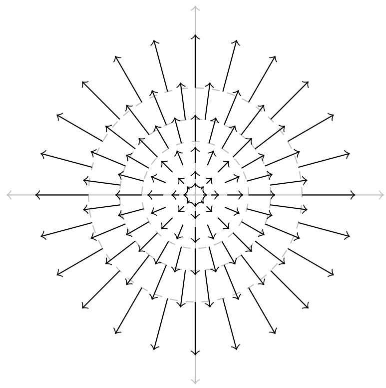

Figure 2.1. The gradient of the potential function $\nabla f = \frac{x^i}{2}\frac{\partial}{\partial x^i}$ for the Gaussian shrinker. Since ∇f points away from the origin, the pullback by ϕ_t expands the metric, which we have to *shrink* to keep the metric static.

2.3.2. Shrinking round spheres.

The metrics of constant positive curvature on the sphere $\mathbb{S}^n$ are naturally shrinking gradient Ricci solitons, when paired with any constant potential function. If $g_{\mathbb{S}^n}$ is the round metric of constant sectional curvature equal to one, the rescaled metric

$$g = 2\,(n-1)\,g_{\mathbb{S}^n} \tag{2.13}$$

will satisfy (2.3) with the canonical choice of constant $\lambda = 1$. For definiteness, we will call $(\mathbb{S}^n, g, n/2)$ the **shrinking round sphere**. (The choice of $f = n/2$ is a convenience that we will explain later.)

The associated self-similar solution is the family $g(t) = (1-t)g$ defined for $t \in (-\infty, 1)$ which simply contracts homothetically as time increases before vanishing identically at $t = 1$. For $t < 1$, the metrics $g(t)$ have radius $r(t) = \sqrt{2(n-1)t}$ and constant sectional curvature $\mathrm{sect}(t) \equiv 1/2(n-1)t$.

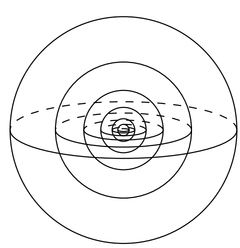

Figure 2.2. A shrinking round sphere.

2.3.3. Einstein manifolds.

The preceding example can be generalized in the following way. To any Einstein manifold $(\mathcal{M}^n, g)$, with

$$\text{Ric} = \frac{\lambda}{2} g, \tag{2.14}$$

of constant scalar curvature $n\lambda/2$, we may naturally associate a Ricci soliton structure of the form $(\mathcal{M}^n, g, f, \lambda)$ of (2.3) with $f = \text{const}$. In particular, every manifold of constant sectional curvature admits a Ricci soliton structure.

If a Ricci soliton $(\mathcal{M}^n, g, X, \lambda)$ is Einstein with constant $\lambda/2$, then

$$\mathcal{L}_X g = \frac{\lambda}{2} g - \text{Ric} = 0; \tag{2.15}$$

i.e., the vector field X is Killing. Thus it is no loss of generality to assume that such an Einstein soliton is gradient relative to a constant potential f. (However, the example of the Gaussian soliton demonstrates that an Einstein manifold may give rise to Ricci soliton structures of more than one type.)

As with the shrinking spheres, the self-similar solutions corresponding to the Einstein solitons evolve purely by scaling. Depending on the sign of λ, the solution $g(t) = (1 - \lambda t)g$ associated to a metric g satisfying (2.14) will shrink, expand, or remain fixed for all t in a maximal interval determined by λ, that is, for all t such that $1 - \lambda t > 0$.

While non-Einstein (a.k.a. **nontrivial**) Ricci solitons will occupy most of our attention, Einstein solitons are nevertheless of fundamental importance in their own right and as building blocks in the construction of other Ricci solitons.

2.3.4. Product solitons.

If $(\mathcal{M}_1^{n_1}, g_1)$ and $(\mathcal{M}_2^{n_2}, g_2)$ are Riemannian manifolds, then the Ricci tensor of the product manifold $(\mathcal{M}_1^{n_1} \times \mathcal{M}_2^{n_2}, g_1 + g_2)$ is itself a product:

$$\text{Ric}(g_1 + g_2) = \text{Ric}(g_1) + \text{Ric}(g_2). \tag{2.16}$$

Here and below, for tensors α_i on $\mathcal{M}_i^{n_i}$, $i = 1, 2$, we will write

$$\alpha_1 + \alpha_2 := p_1^*(\alpha_1) + p_2^*(\alpha_2), \tag{2.17}$$

where $p_i : \mathcal{M}_1^{n_1} \times \mathcal{M}_2^{n_2} \to \mathcal{M}_i^{n_i}$ denotes the projection map. It follows that if $(\mathcal{M}_1^{n_1}, g_1, f_1, \lambda)$ and $(\mathcal{M}_2^{n_2}, g_2, , f_2, \lambda)$ are gradient Ricci soliton structures on $\mathcal{M}_1^{n_1}$ and $\mathcal{M}_2^{n_2}$, respectively, then

$$(\mathcal{M}_1^{n_1} \times \mathcal{M}_2^{n_2}, g_1 + g_2, f_1 + f_2, \lambda) \tag{2.18}$$

is a gradient Ricci soliton structure on $\mathcal{M}_1^{n_1} \times \mathcal{M}_2^{n_2}$. More generally, given two Ricci soliton structures $(\mathcal{M}_i^{n_i}, g_i, X_i, \lambda)$ on $\mathcal{M}_i^{n_i}$, $i = 1, 2$, we have that $(\mathcal{M}_1^{n_1} \times \mathcal{M}_2^{n_2}, g_1 + g_2, (X_1, X_2), \lambda)$ is a Ricci soliton structure on $\mathcal{M}_1^{n_1} \times \mathcal{M}_2^{n_2}$.

For instance, by taking the product of the Gaussian shrinker with the shrinking round sphere of dimension $k \geq 2$, we obtain the **round-cylindrical shrinkers** $\left(\mathbb{S}^k \times \mathbb{R}^{n-k}, g_{\mathrm{cyl}}, f_{\mathrm{cyl}}, 1\right)$, $n \geq 3$, where

$$g_{\mathrm{cyl}} := 2\,(k-1)\, g_{\mathbb{S}^k} + g_{\mathrm{Euc}} \quad \text{and} \quad f_{\mathrm{cyl}}\,(\theta, z) := \frac{|z|^2}{4} + \frac{k}{2}.$$

Here, $|z|^2 = \sum_{i=1}^{n-k} (z^i)^2$, where $z = (z^1, \ldots, z^{n-k}) \in \mathbb{R}^{n-k}$ and $\theta \in \mathbb{S}^k$. The shrinking cylindrical solutions that these Ricci solitons define are of paramount importance in the analysis of singularities of the Ricci flow.

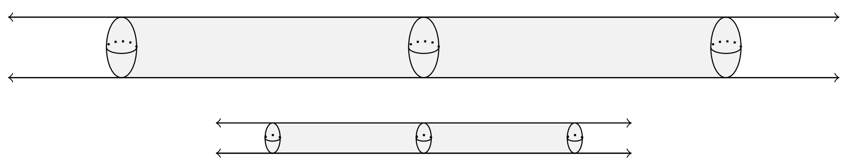

Figure 2.3. *Top*: The shrinker $(\mathbb{S}^{n-1} \times \mathbb{R}^1, g_{\mathrm{cyl}}, f_{\mathrm{cyl}})$. The $\mathbb{S}^{n-1}$ factor is normalized so that its Ricci curvatures are equal to $\frac{1}{2}$.
Bottom: The same shrinker at half the scale.[4] The shading is to indicate the homothetic correspondence. Note however that this is not the correspondence under Ricci flow without diffeomorphism pullback, which shrinks the spheres but not the line.

2.3.5. Quotient solitons.

We will say that a subgroup $\Gamma \subset \mathrm{Isom}(\mathcal{M}^n, g)$ **preserves** the Ricci soliton structure $(\mathcal{M}^n, g, X, \lambda)$ if $\gamma^*(X) = X$ for all $\gamma \in \Gamma$, and preserves the gradient Ricci soliton structure $(\mathcal{M}^n, g, f, \lambda)$ if furthermore $f \circ \gamma = f$ for all $\gamma \in \Gamma$. If Γ is discrete and acts freely and properly discontinuously on $\mathcal{M}^n$, then g and X (respectively, f) descend uniquely to smooth representatives g_{quo} and X_{quo} (respectively, f_{quo}) on the quotient manifold $\mathcal{M}^n/\Gamma$ and define a Ricci soliton structure there.

Example 2.3. The involution $(\theta, r) \mapsto (-\theta, -r)$ on $\mathbb{S}^{n-1} \times \mathbb{R}$ defines a $\mathbb{Z}_2$-quotient of the round-cylindrical shrinker $\left(\mathbb{S}^{n-1} \times \mathbb{R}, g_{\mathrm{cyl}}, f_{\mathrm{cyl}}\right)$. Here, the underlying manifold is diffeomorphic to a nontrivial real line bundle over $\mathbb{RP}^{n-1}$.

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[4]That is, the metric of the bottom cylinder is, up to isometry, equal to $\frac{1}{4}$ times the metric of the top cylinder.

The construction in Example 2.3 can be rephrased in the language of covering spaces. Given a covering space $\pi : \widetilde{\mathcal{M}}^n \to \mathcal{M}^n$ and a Ricci soliton structure $(\mathcal{M}^n, g, X, \lambda)$ on $\mathcal{M}^n$, defining $\tilde{g} = \pi^* g$ and $\tilde{X} = \pi^* X$ yields a Ricci soliton structure on the cover $\widetilde{\mathcal{M}}^n$. If $\pi_1(\widetilde{\mathcal{M}}^n) = \{e\}$, we call this structure the **universal covering soliton**.

2.3.6. Nongradient solitons.

The examples we have considered to this point have all been gradient Ricci solitons. They are the most important kind of Ricci soliton from the perspective of singularity analysis, and all examples which have arisen organically thus as a byproduct of this analysis have proven to be gradient. For example, according to [**245, 251**], any complete shrinking Ricci soliton $(\mathcal{M}^n, g, X, 1)$ of bounded curvature is gradient.

Nevertheless, there are several constructions of nongradient Ricci solitons in the literature and there is no reason to suspect that they are particularly uncommon. Before we give a nontrivial example, let us first describe a superficial means of creating nongradient Ricci solitons from gradient structures. If $(\mathcal{M}^n, g, f, \lambda)$ is a gradient Ricci soliton and $(\mathcal{M}^n, g)$ admits a nontrivial (i.e., not identically zero) Killing vector field K, then adding K to ∇f yields another Ricci soliton structure $(\mathcal{M}^n, g, \nabla f + K, \lambda)$ which will be nongradient provided K is not itself the gradient of a smooth function. Of course this new structure is equivalent to the original one and thus is in a sense "secretly" a gradient Ricci soliton.

The following explicit example of a "true" nongradient Ricci soliton is due to Topping and Yin [**278**].

Example 2.4. The complete Riemannian metric

$$g = \frac{2}{1+y^2}(dx^2 + dy^2), \tag{2.19}$$

together with the complete vector field

$$X = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} \tag{2.20}$$

generated by homotheties, comprises a complete nongradient expanding Ricci soliton structure $(\mathbb{R}^2, g, X, -1)$ on $\mathbb{R}^2$. A short computation shows that the scalar curvature of g is given by (see Figure 2.4)

$$R(x, y) = \frac{1 - y^2}{1 + y^2}. \tag{2.21}$$

Indeed, this follows from (1.20):

$$R_{e^u g_{\mathbb{E}}} = -e^{-u}\Delta u, \tag{2.22}$$

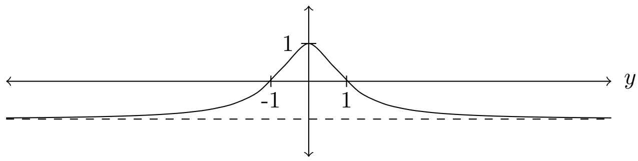

Figure 2.4. The scalar curvature as a function of y: $R(\cdot, y) = \frac{1-y^2}{1+y^2}$.

with $u = \ln\left(\frac{2}{1+y^2}\right)$, and where Δ is the Euclidean Laplacian. We also note that the geometry of $(\mathbb{R}^2, g)$ resembles that of hyperbolic space (with constant sectional curvature $-\frac{1}{2}$) near spatial infinity.

That $(\mathbb{R}^2, g, X, -1)$ is not equivalent to a gradient Ricci soliton structure can be seen by first observing that the Killing vector fields of g are precisely the constant multiples of the vector $\frac{\partial}{\partial x}$.

As we will see below, for any gradient Ricci soliton $(\mathcal{M}^2, g, f, \lambda)$ on an oriented Riemannian surface, the vector $J(\nabla f)$ will be Killing (see Lemma 3.1). Here,

$$J : T\mathcal{M} \to T\mathcal{M} \tag{2.23}$$

is the almost complex structure defined by the conformal class of g and the orientation on $\mathcal{M}^2$, so J is counterclockwise rotation by 90 degrees and $J^2 = -\operatorname{id}_{T\mathcal{M}}$, but for no $c \in \mathbb{R}$ is $J\big(X + c\frac{\partial}{\partial x}\big)$ a constant multiple of $\frac{\partial}{\partial x}$.[5]

Other nontrivial examples of nongradient expanding Ricci solitons can be found in Lott [**223**] and Baird and Danielo [**12, 13**].

2.4. The gradient Ricci soliton equation

In this section we consider basic properties of gradient Ricci solitons in all dimensions. The basic definitions and derived equations were given by Hamilton in various papers, especially [**175, 176, 179**].

2.4.1. Definitions.

Recall from (2.3) that a *gradient Ricci soliton* is a quadruple $(\mathcal{M}^n, g, f, \lambda)$, where $\lambda \in \mathbb{R}$, satisfying

$$\operatorname{Ric} + \nabla^2 f = \frac{\lambda}{2} g, \tag{2.24}$$

where, by Definition 2.1, the *expansion constant* $\lambda > 0$, $= 0$, and < 0 (e.g., $\lambda = 1$, 0, and -1) corresponds to being a *shrinking*, *steady*, and *expanding* gradient Ricci soliton, respectively.

[5]The counterclockwise rotation by 90 degrees J is characterized by the orthonormal frame $\{U, J(U)\}$ being positively oriented for any unit tangent vector U.

Recall that in all cases, f is called the *potential function*. Evident in the above equations is that there should be some relationships between the geometry of g and the analysis of f. Techniques from Ricci flow also prove to be useful. These themes are prevalent throughout this book.

Recall that the Lie derivative of a k-tensor T on a differentiable manifold $\mathcal{M}^n$ satisfies

(2.25)

$$(\mathcal{L}_X T)(Y_1, \ldots, Y_k) = X(T(Y_1, \ldots, Y_k)) - \sum_{i=1}^{k} T(Y_1, \ldots, [X, Y_i], \ldots, Y_k),$$

where $X, Y_1, \ldots, Y_k$ are vector fields. In the case where we are on a Riemannian manifold $(\mathcal{M}^n, g)$, we may re-express this formula in terms of the covariant derivative of g as

(2.26)

$$(\mathcal{L}_X T)(Y_1, \ldots, Y_k) = (\nabla_X T)(Y_1, \ldots, Y_k) + \sum_{i=1}^{k} T(Y_1, \ldots, \nabla_{Y_i} X, \ldots, Y_k).$$

In particular, if T is a 2-tensor, then in local coordinates we have[6]

$$(\mathcal{L}_X T)_{ij} = (\nabla_X T)_{ij} + \nabla_i X^k T_{kj} + \nabla_j X^k T_{ik}. \tag{2.27}$$

Here and throughout the book we use the Einstein summation convention. Notably, (2.25) yields

$$\mathcal{L}_{\nabla f} g = 2\nabla^2 f \tag{2.28}$$

and we may rewrite the gradient Ricci soliton equation (2.24) in terms of the Lie derivative as

$$-2\operatorname{Ric} = \mathcal{L}_{\nabla f} g - \lambda g. \tag{2.29}$$

The left-hand side of this equation is the velocity tensor for Hamilton's **Ricci flow**. Equation (2.29) is an **underdetermined system** of PDEs for the pair (g, f)—there are $\frac{n(n+1)}{2}$ equations for $\frac{n(n+1)}{2} + 1$ unknowns. The Lie derivative term represents the infinitesimal action of the diffeomorphism group on the metric by pullback. A consequence of this is the time-dependent Ricci flow form of a gradient Ricci soliton discussed in Proposition 2.2.

As we shall see, the analysis of (2.29) generally uses techniques from elliptic and parabolic partial differential equations, from the comparison geometry of Ricci curvature, and from Ricci flow. Although we cannot decouple the two quantities g and f, it is often useful to consider the gradient Ricci soliton equation from the point of view of one quantity or the other.

[6]For the reader unfamiliar with local coordinate calculations, Eisenhart's book [**143**] is an excellent classical reference.

Recall that we have the more general notion of *Ricci soliton* $(\mathcal{M}^n, g, X, \lambda)$, where X is a vector field, satisfying

$$2\,\mathrm{Ric} + \mathcal{L}_X g = \lambda g. \tag{2.30}$$

This is also an underdetermined system. In local coordinates,

$$2R_{ij} + \nabla_i X_j + \nabla_j X_i = \lambda g_{ij}. \tag{2.31}$$

Recall that tracing this yields (2.2):

$$R + \operatorname{div} X = \frac{n\lambda}{2}.$$

Observe that if $\mathcal{M}^n$ is closed, then by integrating this and using the divergence theorem, we obtain that the average scalar curvature satisfies

$$R_{\text{avg}} := \frac{\int_{\mathcal{M}} R d\mu}{\mathrm{Vol}(g)} = \frac{n\lambda}{2}, \tag{2.32}$$

where $d\mu$ is the volume form of g and $\mathrm{Vol}(g)$ is the volume of $(\mathcal{M}^n, g)$.

2.5. Product and rotationally symmetric solitons

In this section we consider product structures in more detail and the extent of uniqueness of the potential function f of gradient Ricci soliton structures $(\mathcal{M}^n, g, f)$ for the Riemannian metric g fixed. We also state the uniqueness theorem for rotationally symmetric steady gradient Ricci solitons and the nonexistence theorem for rotationally symmetric shrinking gradient Ricci solitons.

2.5.1. Metric products are soliton products.

If a gradient Ricci soliton is a product metrically, then it is a product as a gradient Ricci soliton.

Lemma 2.5. *Suppose that $(\mathcal{M}^n, g, f, \lambda)$ is a gradient Ricci soliton and that $(\mathcal{M}^n, g)$ is isometric to a Riemannian product $(\mathcal{M}_1^{n_1}, g_1) \times (\mathcal{M}_2^{n_2}, g_2)$. Then for any $x_2 \in \mathcal{M}_2^{n_2}$ we have that $(\mathcal{M}_1^{n_1}, g_1, f_1, \lambda)$ is a gradient Ricci soliton, where $f_1 : \mathcal{M}_1^{n_1} \to \mathbb{R}$ is the restriction of f to $\mathcal{M}_1^{n_1} \times \{x_2\} \cong \mathcal{M}_1^{n_1}$. Of course, the same is true for the roles of the indices 1 and 2 switched.*

Proof. Since $g = g_1 + g_2$, we have for $X, Y \in T\mathcal{M}_1 \cong T(\mathcal{M}_1^{n_1} \times \{x_2\}) \subset T\mathcal{M}$,

$$\begin{aligned}\left(\nabla_g^2 f\right)(X,Y) &= X\,(Yf) - \left\langle \nabla_X^g Y, \nabla f\right\rangle_g \\ &= X\,(Yf) - \left\langle \nabla_X^{g_1} Y, \nabla f_1\right\rangle_{g_1} \\ &= \left(\nabla_{g_1}^2 f_1\right)(X,Y)\end{aligned}$$

because $\nabla^g_X Y = \nabla^{g_1}_X Y$ is tangential to $\mathcal{M}_1^{n_1} \times \{x_2\}$. Therefore, taking the components of $\mathrm{Ric}_g + \nabla^2_g f = \frac{\lambda}{2} g$ in the $\mathcal{M}_1^{n_1}$ directions yields

$$\mathrm{Ric}_{g_1} + \nabla^2_{g_1} f_1 = \frac{\lambda}{2} g_1. \qquad \square$$

2.5.2. Uniqueness and nonuniqueness of the potential function.

Regarding the uniqueness of the potential function of a gradient Ricci soliton with a given metric and a given expansion factor, we have the following.

Proposition 2.6. *Suppose that $(\mathcal{M}^n, g, \lambda)$, with either f_1 or f_2 as its potential function, is a gradient Ricci soliton. Then:*

(1) $f_1 - f_2$ is a constant or

(2) $(\mathcal{M}^n, g)$ is isometric to $(\mathbb{R}, ds^2) \times (\mathcal{N}^{n-1}, h)$, where $(\mathcal{N}^{n-1}, h)$ is isometric to each level set $\{f_1 - f_2 = c\}$ for $c \in \mathbb{R}$.

Moreover, in the second case, $f_1 - f_2$ is linear on the $\mathbb{R}$ factor; that is,

$$f_2(s, x) = f_1(s, x) + as + b \quad \text{for } s \in \mathbb{R},\ x \in \mathcal{N}^{n-1}, \tag{2.33}$$

where $a, b \in \mathbb{R}$.

Proof. Define $F : \mathcal{M}^n \to \mathbb{R}$ by $F := f_1 - f_2$. Then $\nabla^2 F = 0$; i.e., $\mathcal{L}_{\nabla F} g = 0$. Assume that F is not a constant. Then $|\nabla F| = a$, where a is a positive constant. Let φ_t, $t \in \mathbb{R}$, be the 1-parameter group of isometries of $(\mathcal{M}^n, g)$ generated by ∇F. We have $F \circ \varphi_t = F + a^2 t$. Let

$$\Sigma_c := \{F = c\}, \tag{2.34}$$

which is a smooth hypersurface with unit normal $\nu = \frac{\nabla F}{|\nabla F|}$ for each $c \in \mathbb{R}$. The **second fundamental form** II of Σ_c vanishes because

$$\mathrm{II}(X, Y) := \langle \nabla_X \nu, Y \rangle = \left\langle \nabla_X \frac{\nabla F}{|\nabla F|}, Y \right\rangle = \frac{\nabla^2 F(X, Y)}{|\nabla F|} = 0 \tag{2.35}$$

for $X, Y \in T\Sigma_c$. Moreover, since $\mathcal{L}_{\nabla F} g = 0$, φ_t maps Σ_c isometrically onto $\Sigma_{c + a^2 t}$. Hence $(\mathcal{M}^n, g)$ is isometric to $(\mathbb{R} \times \mathcal{N}^{n-1}, a^{-2} dF^2 + h)$, where $(\mathcal{N}^{n-1}, h)$ is isometric to each level set $\{F = c\}$. The proposition follows. $\square$

Remark 2.7. To see the nonuniqueness of the potential function in the splitting case, consider the product of an $(n-1)$-dimensional gradient Ricci soliton $(\mathcal{M}^n, g, f, \lambda)$ with $(\mathbb{R}, ds^2, f_a, \lambda)$, where $f_a(s) = \frac{\lambda}{4}(s - a)^2$ and $a \in \mathbb{R}$.

Corollary 2.8. *If $(\mathcal{M}^n, g, f, \lambda)$ is a gradient Ricci soliton, where $(\mathcal{M}^n, g)$ is equal (isometric) to $(\mathcal{M}_1^{n_1}, g_1) \times (\mathcal{M}_2^{n_2}, g_2)$, then there are $f_i : \mathcal{M}_i^{n_i} \to \mathbb{R}$ such that the $(\mathcal{M}_i^{n_i}, g_i, f_i, \lambda)$ are gradient Ricci solitons, where $f = f_1 + f_2$, or $(\mathcal{M}^n, g)$ splits off an $\mathbb{R}$ factor and $f - f_1 + f_2$ is linear on that $\mathbb{R}$-factor.*

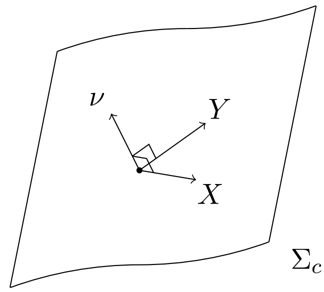

Figure 2.5. A level surface Σ_c of f, a unit normal vector ν to Σ_c, and tangent vectors X, Y to Σ_c.

Proof. Define $f_i : \mathcal{M}_i^{n_i} \to \mathbb{R}$ by Lemma 2.5, so that the $(\mathcal{M}_i^{n_i}, g_i, f_i, \lambda)$ are gradient Ricci solitons. By Proposition 2.6, if $(\mathcal{M}^n, g)$ does not split off an $\mathbb{R}$-factor, then the difference of f and $f_1 + f_2$ is a constant function on $\mathcal{M}^n$, so we may add a constant to, say, f_1 to make them equal. □

If the expansion constants of the gradient Ricci solitons are different, then we have the following.

Proposition 2.9 (GRS that are metrically the same but have different expansion constants)**.** *Suppose that $(\mathcal{M}^n, g)$, with either (f_1, λ_1) or (f_2, λ_2), is a gradient Ricci soliton, where $\lambda_1 \neq \lambda_2$. Then the $(\mathcal{M}^n, g, f_i, \lambda_i)$, for $i = 1, 2$, are both Gaussian solitons.*

Proof. Without loss of generality, we may assume that $\lambda_1 > \lambda_2$. Define $\psi = f_1 - f_2$. Then

$$\nabla^2 \psi = cg, \tag{2.36}$$

where $c := \frac{\lambda_1 - \lambda_2}{2} > 0$. Choose any $p \in \mathcal{M}^n$. Let $\gamma : [0, L] \to \mathcal{M}^n$ be a unit speed geodesic emanating from p and let $\psi(s) := \psi(\gamma(s))$. Then $\psi'(0) \geq -|\nabla\psi|(p)$. Hence $\psi''(s) = c$ implies that

$$\psi(s) \geq \frac{c}{2}s^2 - |\nabla\psi|(p)s + \psi(p) \geq -\frac{1}{2c}|\nabla\psi|^2(p) + \psi(p).$$

This implies that ψ attains its minimum value at some point—call this point $o \in \mathcal{M}^n$—which is unique since ψ is strictly convex. Without loss of generality, we may assume that this minimum value is equal to 0. Hence $\psi > 0$ on $\mathcal{M}^n \setminus \{o\}$.

Now, (2.36) implies that

$$\nabla|\nabla\psi|^2 = 2\nabla^2\psi(\nabla\psi) = 2cg(\nabla\psi) = 2c\nabla\psi.$$

Thus, $|\nabla\psi|^2 = 2c\psi + C$, where C is a constant. Since the minimum of ψ is equal to 0, we have that $C = 0$, so that

$$|\nabla\psi|^2 = 2c\psi. \tag{2.37}$$

Define $\rho := \sqrt{\psi}$. Then

$$|\nabla\rho|^2 = \frac{c}{2} \tag{2.38}$$

on $\mathcal{M}^n \setminus \{o\}$. Moreover, $\nabla(\rho^2) = \nabla\psi$ is a complete vector field which generates a 1-parameter group $\{\varphi_t\}_{t\in\mathbb{R}}$ of homotheties of g. We have that

$$\nabla_{\nabla\rho}(\nabla\rho) = \frac{1}{2}\nabla|\nabla\rho|^2 = 0,$$

where $\nabla\rho$ denotes the gradient of ρ, so that the integral curves to $\nabla\rho$ are geodesics. By Morse theory we have that $\Sigma_t := \rho^{-1}(t)$ is diffeomorphic to $\mathbb{S}^{n-1}$ for all $t \in (0,\infty)$. Since $|\nabla\rho| = 1$, each homothety φ_t of g maps level sets of ρ to level sets of ρ. Hence g may be written as the warped product

$$g = d\rho^2 + \rho^2\tilde{g}, \quad \text{where } \tilde{g} = g|_{\Sigma_1}.$$

Since g is smooth at o, where $\rho = 0$, we have that $(\Sigma_1, \tilde{g})$ must be isometric to the unit $(n-1)$-sphere. Since $\bigcup_{t\in(0,\infty)} \Sigma_t = \mathcal{M}^n \setminus \{o\}$, we conclude that $(\mathcal{M}^n, g)$ is isometric to Euclidean space. The proposition follows. □

Remark 2.10. Compare this to Obata's theorem (see [**249**]), which says that if $(\mathcal{M}^n, g)$ is a complete Riemannian manifold with a nonconstant function f satisfying $\nabla^2 f = -fg$, then $(\mathcal{M}^n, g)$ is isometric to the unit n-sphere.

Note that from the equality case of Theorem 2.14 below, we have that a flat shrinking gradient Ricci soliton must be the Gaussian shrinking gradient Ricci soliton.

2.5.3. Uniqueness of rotationally symmetric gradient Ricci solitons.

We have the following uniqueness result, due to Bryant [**54**] in the steady case and Kotschwar [**204**] in the shrinking case.

Theorem 2.11.

(1) *Any complete rotationally symmetric steady gradient Ricci soliton must be flat or the Bryant soliton.*

(2) *Any complete rotationally symmetric shrinking gradient Ricci soliton must be the Gaussian shrinking gradient Ricci soliton on $\mathbb{R}^n$, the round cylinder shrinker on $\mathbb{S}^{n-1}\times\mathbb{R}$, or the round sphere shrinker on $\mathbb{S}^n$.*

Assuming nonflatness, the idea of the proof is to first show that the potential function is rotationally symmetric (see Exercise 6.2 below). The gradient Ricci soliton equation is a nonlinear second-order ODE, which may be then reduced to a first-order system of ODEs. An ODE analysis using the metric's smoothness at any finite end (removable singularity) and completeness at any infinite end yields the classification. A detailed proof of Theorem 2.11(1), with calculations related to the proof of Theorem 2.11(2), will be given in Chapter 6.

Remark 2.12. For an exposition of Bryant's work on rotationally symmetric *expanding* gradient Ricci solitons, see §5 of Chapter 1 in [**101**]. We summarize the results in §7.1.2 of this book.

2.6. Fundamental identities: Differentiating the Ricci soliton equation

In this section we present basic identities satisfied by gradient Ricci solitons. These identities are fundamental to the study of gradient Ricci solitons.

2.6.1. Trace and divergence of the gradient Ricci soliton equation.

Let $(\mathcal{M}^n, g, f, \lambda)$ be a gradient Ricci soliton. By tracing the gradient Ricci soliton equation (2.24), we obtain

$$R + \Delta f = \frac{n\lambda}{2}. \tag{2.39}$$

On the other hand, taking the divergence of (2.24) while applying the following contracted second Bianchi identity (1.60) yields

$$\frac{1}{2} dR + \Delta (df) = 0.$$

By the commutator formula (1.52), for any function u and by (2.39), we have

$$0 = \frac{1}{2} dR + d(\Delta f) + \operatorname{Ric}(\nabla f) = -\frac{1}{2} dR + \operatorname{Ric}(\nabla f).$$

We write this as the following basic equation:

$$2 \operatorname{Ric}(\nabla f) = \nabla R. \tag{2.40}$$

A useful consequence of this is

$$\langle \nabla f, \nabla R \rangle = 2 \operatorname{Ric}(\nabla f, \nabla f). \tag{2.41}$$

2.6.2. A fundamental identity relating R and f.

Now by (2.24), for any vector field V,

$$\begin{aligned} V(|df|^2) &= 2 \langle \nabla_V df, df \rangle \\ &= 2 \left\langle -\operatorname{Ric}(V) + \frac{\lambda}{2} g(V), df \right\rangle \\ &= (-2 \operatorname{Ric}(\nabla f) + \lambda df)(V), \end{aligned}$$

so that

$$\nabla |\nabla f|^2 = -2 \operatorname{Ric}(\nabla f) + \lambda \nabla f. \tag{2.42}$$

Combining this with (2.40) yields

$$\nabla(R + |\nabla f|^2 - \lambda f) = 0. \tag{2.43}$$

Since $\mathcal{M}^n$ is connected, we conclude that

$$R + |\nabla f|^2 - \lambda f = C, \tag{2.44}$$

where C is a constant. This equation is used in a fundamental way to understand gradient Ricci solitons. The above equations were obtained by Hamilton.

If $\lambda = \pm 1$ (shrinking or expanding gradient Ricci soliton), then by adding a constant to the potential function f we may assume that $C = 0$, so that

$$R + |\nabla f|^2 = \lambda f. \tag{2.45}$$

If $\lambda = 0$ (steady gradient Ricci soliton) and g is not Ricci-flat, then by scaling the metric we may take $C = 1$, so that

$$R + |\nabla f|^2 = 1. \tag{2.46}$$

In other words, we may choose $C = 1 - |\lambda|$. In these cases we say that the gradient Ricci soliton is a **normalized gradient Ricci soliton**. Throughout this book, unless otherwise indicated we shall always assume that we are on a normalized gradient Ricci soliton.

2.6.3. The f-scalar curvature and f-Ricci tensor.

Define the **f-scalar curvature** to be

$$R_f := R + 2\Delta f - |\nabla f|^2 . \tag{2.47}$$

We define the **f-Ricci tensor**, a.k.a. the **Bakry–Emery tensor**, by

$$\operatorname{Ric}_f = \operatorname{Ric} + \nabla^2 f.$$

Then the gradient Ricci soliton equation is

$$\operatorname{Ric}_f = \frac{\lambda}{2} g. \tag{2.48}$$

Remark 2.13. From (2.39), (2.45), and (2.46), on a (normalized) gradient Ricci soliton we have

$$R_f = -\lambda f + n\lambda - 1 + |\lambda|. \tag{2.49}$$

2.6.4. f-Laplacian-type equations.

Define the f**-Laplacian** by

$$\Delta_f := \Delta - \nabla f \cdot \nabla. \tag{2.50}$$

This natural elliptic operator is prevalent in computations regarding gradient Ricci solitons. For any functions $A, B : \mathcal{M}^n \to \mathbb{R}$, provided we can integrate by parts (e.g., if A and B have compact support), we have

$$\int_{\mathcal{M}} A \Delta_f B \, e^{-f} d\mu = -\int_{\mathcal{M}} \langle \nabla A, \nabla B \rangle \, \mathrm{e}^{-f} d\mu = \int_{\mathcal{M}} B \Delta_f A \, e^{-f} d\mu. \tag{2.51}$$

That is, the operator Δ_f is formally **self-adjoint** on $L^2(\mathrm{e}^{-f} d\mu)$. Moreover, for any $\varphi : \mathcal{M}^n \to \mathbb{R}$ we have that

$$\left(\Delta_f - \frac{1}{4} R_f \right) \varphi = e^{f/2} \left(\Delta - \frac{1}{4} R \right) (\mathrm{e}^{-f/2} \varphi). \tag{2.52}$$

By (2.45) and (2.46) and by their differences with (2.39), we obtain the following for each of the three types of normalized gradient Ricci solitons.

(1) For a shrinking gradient Ricci soliton, we have

$$R + |\nabla f|^2 = f, \quad \text{so that } R \leq f, \tag{2.53}$$

and

$$\Delta_f f = \frac{n}{2} - f. \tag{2.54}$$

Hence $f - \frac{n}{2}$ is an eigenfunction of $-\Delta_f$ with eigenvalue 1.

(2) For a non-Ricci-flat steady gradient Ricci soliton, we have

$$R + |\nabla f|^2 = 1, \quad \text{so that } R \leq 1, \tag{2.55}$$

and

$$\Delta_f f = -1. \tag{2.56}$$

(3) For an expanding gradient Ricci soliton, we have

$$R + |\nabla f|^2 = -f, \quad \text{so that } R \leq -f, \tag{2.57}$$

and

$$\Delta_f f = f - \frac{n}{2}. \tag{2.58}$$

By taking the divergence of (2.40) and then applying (1.60) and (2.24), we obtain

$$\begin{aligned} \Delta R &= 2 \operatorname{div}(\mathrm{Ric})(\nabla f) + 2 \langle \mathrm{Ric}, \nabla^2 f \rangle \\ &= \langle \nabla R, \nabla f \rangle - 2 \left\langle \mathrm{Ric}, \mathrm{Ric} - \frac{\lambda}{2} g \right\rangle. \end{aligned} \tag{2.59}$$

That is,

$$\Delta_f R = -2\,|\mathrm{Ric}|^2 + \lambda R. \tag{2.60}$$

Thus

$$\Delta_f R \leq -\frac{2}{n} R^2 + \lambda R. \tag{2.61}$$

It is convenient to define the **f-divergence**

$$\mathrm{div}_f(T) = \mathrm{div}(T) - \mathrm{tr}^{1,2}(\nabla f \otimes T) = (\mathrm{div} - \iota_{\nabla f})(T) = e^f \,\mathrm{div}(\mathrm{e}^{-f} T) \tag{2.62}$$

acting on tensors, where $\mathrm{tr}^{a,b}$ denotes the trace over the ath and bth components. For example,

$$\Delta_f u = \mathrm{div}_f(du) = \mathrm{div}_f(\nabla u).$$

2.7. Sharp lower bounds for the scalar curvature

2.7.1. Statements and consequences of the lower bounds.

We have seen that every Einstein manifold admits at least one Ricci soliton structure and that these are precisely the Ricci soliton structures of constant scalar curvature. The following theorem shows that the scalar curvature of *any* complete Ricci soliton is bounded from below by a sharp constant. This follows in the gradient case from the work of B.-L. Chen [**86**] on ancient solutions and from the work of Z.-H. Zhang [**303**] on GRS. The equality case when $\lambda > 0$ is due to Pigola, Rimoldi, and Setti [**258**].

Theorem 2.14 (Sharp scalar curvature lower bounds for Ricci solitons)**.** *If $(\mathcal{M}^n, g, X, \lambda)$ is a complete Ricci soliton, then:*

(a) $R \geq 0$ *if* $\lambda \geq 0$.

(b) $R \geq \frac{\lambda n}{2}$ *if* $\lambda < 0$.

Moreover, if equality holds at any point of $\mathcal{M}^n$, then $(\mathcal{M}^n, g)$ is Einstein. If $\lambda > 0$ and the shrinker is gradient, that is, $X = \nabla f$ for some function f, with $R = 0$ at some point, then $(\mathcal{M}^n, g, f)$ is a Gaussian shrinker.

Before proving this, we observe that Theorem 2.14 yields a measure of control of the potential function:

Corollary 2.15 (Potential function estimates)**.** *Let $(\mathcal{M}^n, g, f, \lambda)$ be a GRS and let $p \in \mathcal{M}^n$.*

(1) *On a shrinking GRS ($\lambda = 1$),*

(2.63)

$$|\nabla f|^2 \leq f, \quad R \leq f, \quad \Delta f \leq \frac{n}{2}, \quad \textit{and} \quad \sqrt{f}\,(x) \leq \sqrt{f}(p) + \frac{1}{2} d(x,p),$$

where $d(x,p)$ denotes the Riemannian distance from x to p with respect to the metric g. At a minimum point[7] *$o \in \mathcal{M}^n$ of f we have $0 \leq R(o) = f(o) \leq \frac{n}{2}$ and*

$$f(x) \leq \frac{1}{4}\left(d(x,o) + \sqrt{2n}\right)^2. \tag{2.64}$$

(2) *On a steady GRS ($\lambda = 0$),*

$$|\nabla f|^2 \leq 1, \quad R \leq 1, \quad \Delta f \leq 0, \quad \text{and} \quad |f(x) - f(p)| \leq d(x,p). \tag{2.65}$$

(3) *On an expanding GRS ($\lambda = -1$),*

$$|\nabla f|^2 \leq \frac{n}{2} - f, \quad \Delta f \leq 0, \quad \text{and} \quad \sqrt{\frac{n}{2} - f(x)} \leq \sqrt{\frac{n}{2} - f(p)} + \frac{1}{2}d(x,p). \tag{2.66}$$

In particular, $f \leq \frac{n}{2}$.

Proof of Corollary 2.15. The upper bounds for Δf follow from (2.39) and Theorem 2.14. The upper bounds for R follow from (2.45) and (2.46). The upper bounds for $|\nabla f|^2$ follow from (2.45), (2.46), and Theorem 2.14. By integrating the bounds for $|\nabla f|$ along minimal geodesics, we obtain the inequalities for f and its square root.

In the case of a shrinking GRS, by (2.54), at a minimum point o of f we have $f(o) - R(o) = |\nabla f|^2(o) = 0$ and

$$0 \leq \Delta_f f(o) = \frac{n}{2} - f(o). \tag{2.67}$$

Thus $0 \leq f(o) = R(o) \leq \frac{n}{2}$. Now, integrating the inequality $|\nabla(2\sqrt{f})| \leq 1$ from Theorem 2.14 yields

$$2\sqrt{f(x)} \leq 2\sqrt{f(o)} + d(x,o) \leq \sqrt{2n} + d(x,o),$$

which in turn implies (2.64). □

2.7.2. Laplacian comparison on Riemannian manifolds.

A basic tool that we will use to prove Theorem 2.14 is the *Laplacian comparison theorem* for the distance function on Riemannian manifolds, which we recall in this subsection.

Let $(\mathcal{M}^n, g)$ be a Riemannian manifold. Recall that the length of a path $\gamma : [a,b] \to \mathcal{M}^n$ is defined by

$$\mathrm{L}(\gamma) := \int_a^b |\gamma'(r)|\,dr. \tag{2.68}$$

[7] We will show in Theorem 4.3 below that the infimum of f over $\mathcal{M}^n$ is attained at some point.

The distance function $d : \mathcal{M}^n \times \mathcal{M}^n \to [0, \infty)$ is defined as an infimum of lengths:

$$d(x,y) = \inf_{\gamma} \mathrm{L}(\gamma), \tag{2.69}$$

where the infimum is taken over all paths joining x and y.

Let $(\mathcal{M}^n, g)$ be a Riemannian manifold. Let $\gamma_v : [0, L] \to \mathcal{M}^n$ be a 1-parameter family of piecewise smooth paths such that $\gamma := \gamma_0$ (but not necessarily γ_v for $v \neq 0$) is parametrized by arc length. Then the *first variation of arc length formula* says (see Exercise 2.22)

$$\left.\frac{d}{dv}\right|_{v=0} \mathrm{L}\left(\gamma_v\right) = -\int_0^L \left\langle V(r), \nabla_{\gamma'(r)}\gamma'(r)\right\rangle dr + \left\langle V(r), \gamma'(r)\right\rangle\Big|_{r=0}^{L}, \tag{2.70}$$

where $V(r) := \left.\frac{\partial}{\partial v}\right|_{v=0} \gamma_v(r)$. In particular, by considering the case where both $V(0) = 0$ and $V(L) = 0$, we see that γ is a critical point of the arc length functional L if and only if $\nabla_{\gamma'(r)}\gamma'(r) \equiv 0$; i.e., γ is a geodesic.

The *second variation of arc length formula* tells us the following (see (1.17) in Cheeger and Ebin's book [**84**]); cf. Exercise 2.23.

Proposition 2.16. *Suppose that $p := \gamma_v(0)$ is independent of v and that $\gamma = \gamma_0$ is a unit speed geodesic. Then the second variation of the length* L *is*

$$\begin{aligned}\left.\frac{d^2}{dv^2}\right|_{v=0} \mathrm{L}\left(\gamma_v\right) = &\int_0^L \left(\left|(\nabla_{\gamma'(r)}V)^{\perp}\right|^2 - \left\langle \mathrm{Rm}(V, \gamma'(r))\gamma'(r), V\right\rangle\right) dr \\ &+ \left\langle \nabla_V\left(\frac{\partial}{\partial v}\gamma_v\right), \gamma'(L)\right\rangle,\end{aligned} \tag{2.71}$$

where $(\nabla_{\gamma'}V)^{\perp} := \nabla_{\gamma'}V - \langle\nabla_{\gamma'}V, \gamma'\rangle\gamma'$ is the projection of $\nabla_{\gamma'}V$ onto the hyperplane $(\gamma')^{\perp} = \{V \in T\mathcal{M} : \langle V, \gamma'\rangle = 0\}$.

We shall also use the notation $\delta_V^2 \mathrm{L}(\gamma) := \left.\frac{\partial^2}{\partial v^2}\right|_{v=0} \mathrm{L}\left(\gamma_v\right)$. Since the distance function is only Lipschitz continuous, when considering its Laplacian we shall use the following.

Definition 2.17. Let $\varphi : \mathcal{M}^n \to \mathbb{R}$ be continuous in a neighborhood of a point x. We say that $\Delta\varphi\,(x) \leq A$ in the **barrier sense** if for any $\varepsilon > 0$ there exists a C^2 function $\psi \geq \varphi$ defined in a neighborhood of x such that $\psi\,(x) = \varphi\,(x)$ and $\Delta\psi\,(x) \leq A + \varepsilon$.

We say that $\Delta\varphi\,(x) \leq A$ in the **strong barrier sense** if there exists a C^2 function $\psi \geq \varphi$ defined in a neighborhood of x such that $\psi\,(x) = \varphi\,(x)$ and $\Delta\psi\,(x) \leq A$. We have the analogous definitions for the operator Δ_f.

Fix $p \in \mathcal{M}^n$ and denote $r(x) := d(x,p)$. Let $r_x := r(x)$. By applying the second variation of arc length formula, we obtain the following upper bound for the Laplacian of the distance function (cf. Li's book [**217**]).

Proposition 2.18. *Let $x \neq p$, let $\gamma : [0, r_x] \to \mathcal{M}^n$ be a unit speed minimal geodesic joining p to x, and let $\zeta : [0, r_x] \to \mathbb{R}$ be a continuous piecewise C^∞ function satisfying $\zeta(0) = 0$ and $\zeta(r_x) = 1$. Then in the strong barrier sense we have*

$$\Delta r(x) \leq \int_0^{r_x} \left((n-1) \left(\zeta'\right)^2 (r) - \zeta^2 (r) \operatorname{Ric}\left(\gamma'(r), \gamma'(r)\right) \right) dr. \tag{2.72}$$

In particular, the above inequality holds in the classical sense if x is not in the cut locus of p.

Proof. Fix $p \in \mathcal{M}^n$ and let $x \neq p$. Let $\varepsilon \in (0, \operatorname{inj}_g(x))$, where $\operatorname{inj}_g(x)$ denotes the injectivity radius of g at x. We extend γ to an n-parameter family of paths by defining $\gamma^V : [0, r_x] \to \mathcal{M}^n$ for $V \in B_\varepsilon(0) \subset T_x\mathcal{M}$ by

$$\gamma^V(r) := \exp_{\gamma(r)}(\zeta(r) V(r)),$$

where $V(r) \in T_{\gamma(r)}\mathcal{M}$ is the parallel transport of V along γ and where $\zeta : [0, r_x] \to \mathbb{R}$ satisfies $\zeta(0) = 0$ and $\zeta(r_x) = 1$. Note that $V(r_x) = V$.

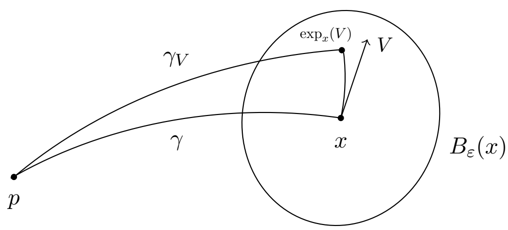

Figure 2.6. A path γ^V, where $V \in B_\varepsilon(0) \subset T_x\mathcal{M}$. Note that γ is a minimal geodesic, but γ_V is not necessarily a geodesic.

The family of paths γ^V have the properties that $\gamma^0(r) = \gamma(r)$, $\gamma^V(0) = p$, $\gamma^V(r_x) = \exp_x(V)$, and

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \gamma^{tV}(r) = \zeta(r) V(r).$$

We have

$$\mathrm{L}\left(\gamma^V\right) \geq r(\exp_x(V)), \tag{2.73a}$$

$$\mathrm{L}\left(\gamma^0\right) = r_x. \tag{2.73b}$$

Since $\varepsilon < \operatorname{inj}_g(x)$, $\exp_x : B_\varepsilon(0) \to B_\varepsilon(x)$ is a diffeomorphism. Let $y \in B_\varepsilon(x)$. Note that $\exp_x^{-1}(y) \in B_\varepsilon(0) \subset T_x\mathcal{M}$. So (2.73) implies that the C^∞ function $\varphi : B_\varepsilon(x) \to \mathbb{R}$ defined by

$$\varphi(y) = \mathrm{L}(\gamma^{\exp_x^{-1}(y)})$$

is an *upper barrier* for r at x; that is, $\varphi(y) \geq r(y)$ for $y \in B_\varepsilon(x)$ and $\varphi(x) = r_x$. Thus, in the strong barrier sense of Definition 2.17, we have

$$\Delta r(x) \leq \Delta\varphi(x). \tag{2.74}$$

Let the vectors $\{e_1, \ldots, e_{n-1}\}$ complete the tangent vector $\gamma'(r_x)$ to an orthonormal basis of $T_x\mathcal{M}$. Then its parallel transport along γ, written as $\{e_1(r), \ldots, e_{n-1}(r), \gamma'(r)\}$, forms an orthonormal basis of $T_{\gamma(r)}\mathcal{M}$ for each $r \in [0, r_x]$. By (2.71), we have

$$\begin{aligned}
\Delta\varphi(x) &= \sum_{i=1}^{n-1} \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \varphi(\exp_x(te_i)) + \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \varphi\left(\exp_x\left(t\gamma'(r_x)\right)\right) \\
&= \sum_{i=1}^{n-1} \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \mathrm{L}\left(\gamma^{te_i}\right) \\
&= \sum_{i=1}^{n-1} \int_0^{r_x} \left(\left(\zeta'\right)^2(r) - \zeta^2(r)\left\langle \mathrm{Rm}(e_i, \gamma'(r))\gamma'(r), e_i\right\rangle\right) dr,
\end{aligned}$$

where we used $\varphi(\exp_x(t\gamma'(r_x))) = r_x + t$ and $\langle \nabla_{e_i} e_i, \gamma'(r_x)\rangle = 0$ (since $\gamma^{te_i}(r_x) = \exp_x(te_i)$ is a geodesic). The proposition follows. $\square$

The proposition leads to the following question: What are good or optimal choices for $\zeta(r)$ in (2.72)? By taking $\zeta(r) = \frac{r}{r_x}$, a choice which for the case of Euclidean space corresponds to variations comprising straight lines, we obtain the Laplacian comparison theorem:

Corollary 2.19. *If* $(\mathcal{M}^n, g)$ *is a complete Riemannian manifold with* $\mathrm{Ric} \geq 0$*, then*

$$\Delta r(x) \leq \frac{n-1}{r(x)} \tag{2.75}$$

in the strong barrier sense.

On the other hand, it is useful to consider a choice of $\zeta(r)$ which corresponds to a frame of parallel unit vector fields except near the ends of the geodesic, where the variations taper down. Now let $x \in \mathcal{M}^n \setminus B_2(p)$ and let $\gamma : [0, r(x)] \to \mathcal{M}^n$ be a unit speed minimal geodesic joining p to x. Define $\zeta : [0, r(x)] \to [0, 1]$ to be the piecewise linear function

$$\zeta(r) = \begin{cases} r & \text{if } 0 \leq r \leq 1, \\ 1 & \text{if } 1 < r \leq r(x) - 1, \\ r(x) - r & \text{if } r(x) - 1 < r \leq r(x). \end{cases} \tag{2.76}$$

Let $\{e_1, \ldots, e_{n-1}, \gamma'(0)\}$ be an orthonormal basis of $T_p\mathcal{M}$. Define $e_i(r) \in T_{\gamma(r)}\mathcal{M}$ to be the parallel transport of $e_i = e_i(0)$ along γ. Then the frame

$\{e_1(r), \ldots, e_{n-1}(r), \gamma'(r)\}$ forms an orthonormal basis of $T_{\gamma(r)}\mathcal{M}$ for $r \in [0, r(x)]$. Since γ is minimal, by the second variation of arc length formula, we have for each i,

$$0 \leq \delta^2_{\zeta e_i} \mathrm{L}(\gamma) = \int_0^{r(x)} \left((\zeta')^2(r) - \zeta^2(r) \left\langle \mathrm{Rm}\left(\gamma'(r), e_i\right) e_i, \gamma'(r) \right\rangle \right) dr.$$

Summing over i, we obtain

$$\int_0^{r(x)} \zeta^2(r) \,\mathrm{Ric}\left(\gamma'(r), \gamma'(r)\right) dr \leq 2(n-1). \tag{2.77}$$

Let

$$\mathrm{S}(x) := \sup_{V \in \mathcal{S}_y^{n-1},\, y \in B_1(x)} \mathrm{Ric}(V, V)_+, \tag{2.78}$$

where $\mathcal{S}_y^{n-1} \subset T_y\mathcal{M}$ is the unit $(n-1)$-sphere. We conclude:

Lemma 2.20. *If $x \in \mathcal{M}^n \setminus B_2(p)$ and if $\gamma : [0, r(x)] \to \mathcal{M}^n$ is a unit speed minimal geodesic joining p to x, then*

$$\int_0^{r(x)} \mathrm{Ric}\left(\gamma'(r), \gamma'(r)\right) dr \leq 2(n-1) + \frac{2}{3}\left(\mathrm{S}(p) + \mathrm{S}(x)\right). \tag{2.79}$$

This lemma estimates, in an integral sense, the amount of positive Ricci curvature in the tangential direction that there can be along a minimal geodesic.

We now apply the Laplacian upper bound (2.72) to prove the following differential inequality for the distance function on Ricci solitons in terms of the **X-Laplacian** operator:

$$\Delta_X \phi := \Delta\phi - \langle X, \nabla\phi \rangle. \tag{2.80}$$

Proposition 2.21. *Let $(\mathcal{M}^n, g, X, \lambda)$ be a complete Ricci soliton and let $r = d(p, \cdot)$ be the distance from a fixed $p \in \mathcal{M}^n$. Suppose that $|\mathrm{Ric}| \leq K_0$ on $B_p(r_0)$. Then there is a constant $C = C(n)$ such that the inequality*

$$\Delta_X r \leq -\frac{\lambda}{2} r + C(n)\left(K_0 r_0 + r_0^{-1}\right) + |X|(p) =: h \tag{2.81}$$

holds in the barrier sense on $\mathcal{M}^n \setminus B_{r_0}(p)$; that is, for every $x \in \mathcal{M}^n \setminus B_{r_0}(p)$ and $\varepsilon > 0$ there exists a C^2 function $\psi \geq r$ defined in a neighborhood of x such that $\psi(x) = r(x)$ and $\Delta\psi(x) \leq h + \varepsilon$.

Proof. Suppose that x is not in the cut locus of p. Since γ is a geodesic, by applying the fundamental theorem of calculus and using the Ricci soliton equation, we obtain

$$\begin{aligned}(2.82)\quad \langle X, \nabla r\rangle(x) - \langle X(p), \gamma'(0)\rangle &= \int_0^{r_x} \frac{d}{dr}\langle X(\gamma(r)), \gamma'(r)\rangle\, dr \\ &= \int_0^{r_x} (\nabla X)(\gamma'(r), \gamma'(r))\, dr \\ &= -\int_0^{r_x} \operatorname{Ric}\left(\gamma'(r), \gamma'(r)\right) dr + \frac{\lambda}{2} r\,(x)\,.\end{aligned}$$

By combining this with (2.72), we obtain

$$\begin{aligned}(2.83)\quad \Delta_X r(x) \leq \int_0^{r_x} &\left((n-1)(\zeta')^2(r) + (1-\zeta^2(r)) \operatorname{Ric}\left(\gamma'(r), \gamma'(r)\right)\right) dr \\ &- \frac{\lambda}{2} r(x) + \langle X(p), \gamma'(0)\rangle.\end{aligned}$$

Let $\zeta(r) = \frac{r}{r_0}$ for $0 \leq r \leq r_0$ and $\zeta(r) = 1$ for $r_0 < r \leq r_x$. We then conclude from (2.83) that

$$\Delta_X r(x) \leq \frac{n-1}{r_0} + \frac{2}{3} r_0 \operatorname{S}(p) - \frac{\lambda}{2} r(x) + |X(p)|,$$

where $\operatorname{S}(p)$ is defined by (2.78). The proposition follows. □

2.7.3. Proof of the scalar curvature lower bound.

We are now ready to prove Theorem 2.14. The argument given in [**303**] for gradient Ricci solitons extends essentially verbatim to the nongradient case; we tweak it slightly to obtain a sharp constant in the expanding case.

The proof will also make use of the following specialized *cutoff function.*

Proposition 2.22. *For each* $0 < \delta < 1/10$*, there exists a smooth function* $\varphi = \varphi_\delta : \mathbb{R} \to [0, 1]$ *such that*

$$(2.84)\qquad \varphi(x) = \begin{cases} 1 & \text{if } x \leq \delta, \\ 0 & \text{if } x \geq 2, \end{cases} \qquad -(1+\theta)\sqrt{\varphi} \leq \varphi' \leq 0, \quad |\varphi''| \leq C_0,$$

and

$$(2.85)\qquad 1 - \varphi(x) + \frac{x}{2}\varphi'(x) \geq -\varepsilon,$$

where $\theta = \theta(\delta)$ *and* $\varepsilon = \varepsilon(\delta)$ *are positive and tend to* 0 *as* $\delta \to 0$.

Proof of Proposition 2.22. Fix any $0 < \delta < 1/10$. We start with a smooth function $\eta = \eta_\delta$ satisfying

$$\eta(x) = \begin{cases} 1 & \text{if } x \in (-\infty, \delta], \\ \frac{2-\delta-x}{2-3\delta} & \text{if } x \in [3\delta, 2-2\delta], \\ 0 & \text{if } x \in [2, \infty) \end{cases}$$

and

$$-\frac{1}{2}(1+\theta) \leq \eta' \leq 0, \quad |\eta''| \leq C_1,$$

where $C_1 = C_1(\delta) > 0$ and $\theta = \theta(\delta) > 0$ tends to 0 as $\delta \to 0$. Thus η is a smooth approximation to the piecewise linear function that is equal to 1 for $x \leq 2\delta$, decreases linearly to 0 over the interval $[2\delta, 2-\delta]$, and is equal to 0 for $x \geq 2-\delta$. Then $\varphi := \eta^2$ satisfies

$$-(1+\theta)\sqrt{\varphi} \leq \varphi' \leq 0 \quad \text{and} \quad |\varphi''| \leq C_0 := 2C_1.$$

To verify (2.85), we only need to consider $x \in [\delta, 2]$. We consider three cases. First, for $x \in [\delta, 3\delta]$, we have

$$1 - \varphi + \frac{x}{2}\varphi' \geq -3\delta|\varphi'| \geq -3\delta(1+\theta).$$

Next, for $x \in [3\delta, 2-2\delta]$,

$$\begin{aligned} 1 - \varphi(x) + \frac{x}{2}\varphi'(x) &= 1 - \eta(x)(\eta(x) - x\eta'(x)) \\ &= 1 - \frac{(2-\delta-x)(2-\delta)}{(2-3\delta)^2} \\ &= \frac{(2-\delta)x - 8\delta + 8\delta^2}{(2-3\delta)^2} \\ &\geq -2\delta. \end{aligned}$$

Finally, for $x \in [2-2\delta, 2]$, since φ is decreasing, we have that $\varphi(x) \leq \delta^2/(2-3\delta)^2 \leq \delta^2$ and therefore

$$1 - \varphi + \frac{x}{2}\varphi' \geq 1 - \delta^2 - (1+\theta)\delta \geq -\theta\delta.$$

Thus φ satisfies (2.85). □

Proof of Theorem 2.14. For the case where $\mathcal{M}^n$ is compact, which is quite easy, see Exercise 2.11.

Let $p \in \mathcal{M}^n$ and define $r(x) = d(x,p)$. Choose $0 \leq r_0 < 1$ such that $|X(p)| \leq r_0^{-1}$ and $|\text{Ric}| \leq r_0^{-2}$ on $B_{r_0}(p)$. For each $0 < \delta < 1/10$ and $a > 1/\delta$, let $\varphi = \varphi_\delta$ be as in Proposition 2.22 and define $\phi = \phi_{\delta,a} : \mathcal{M}^n \to [0,1]$ by

$$\phi(x) = \varphi(r(x)/(ar_0)).$$

Let x_0 be a point at which the compactly supported function

$$F := F_{\delta,a} := \phi_{\delta,a} R : \mathcal{M}^n \to \mathbb{R} \tag{2.86}$$

achieves its minimum value. We claim that

$$F(x_0) \geq \begin{cases} -C_1/a & \text{if } \lambda \geq 0, \\ (1+\varepsilon)\frac{n\lambda}{2} - \frac{C_1}{a} & \text{if } \lambda < 0, \end{cases} \tag{2.87}$$

where $C_1 = C_1(n, \delta, \lambda, r_0)$ is a positive constant independent of a and $\varepsilon = \varepsilon(\delta)$ is positive and tends to 0 as $\delta \to 0$.

To see this, first consider the case that $x_0 \in B_{\delta a r_0}(p)$. Then $F \equiv R$ in a neighborhood of x_0 and

$$0 \leq \Delta_X F = \Delta_X R = -2|\mathrm{Ric}|^2 + \lambda R = -2\left|\mathrm{Ric} - \frac{R}{n} g\right|^2 - \frac{2}{n} R \left(R - \frac{n\lambda}{2} \right) \tag{2.88}$$

at x_0, where the second equality is by Exercise 2.30. Since the first term is nonpositive, the second term must be nonnegative. So $F(x_0) = R(x_0) \geq 0$ if $\lambda \geq 0$ and $F(x_0) = R(x_0) \geq n\lambda/2$ if $\lambda < 0$. Either way, (2.87) holds in this situation.

Now suppose that $x_0 \notin B_{\delta a r_0}(p)$. If $F(x_0) \geq 0$, then (2.87) holds and there is nothing to prove, so we may assume that $F(x_0) < 0$. In particular, $x_0 \in B_{2ar_0}(p)$ and $\phi(x_0) > 0$. By Calabi's trick[8], we may assume r is smooth at x_0 and compute that

$$\begin{aligned} 0 &\leq \Delta_X F \\ &= \phi \Delta_X R + 2\langle \nabla R, \nabla \phi \rangle + R \Delta_X \phi \\ &\leq -\frac{2F}{n}\left(R - \frac{n\lambda}{2} \right) - 2R \frac{|\nabla \phi|^2}{\phi} + R\Delta_X \phi. \end{aligned} \tag{2.89}$$

Here, we have used that $\nabla R = -R\nabla\phi/\phi$ at x_0, since $\nabla F(x_0) = 0$. By Proposition 2.21 and our choice of r_0, we have

$$\Delta_X r \leq \begin{cases} C(n)/r_0 & \text{if } \lambda \geq 0, \\ C(n)/r_0 - \frac{\lambda}{2} r & \text{if } \lambda < 0, \end{cases} \tag{2.90}$$

and hence

$$\Delta_X \phi = \frac{\varphi'}{ar_0}\Delta_X r + \frac{\varphi''}{a^2 r_0^2} \geq \begin{cases} -\frac{C_2}{a} & \text{if } \lambda \geq 0, \\ \frac{\lambda r \varphi'}{2ar_0} - \frac{C_2}{a} & \text{if } \lambda < 0, \end{cases} \tag{2.91}$$

for some constant $C_2 = C_2(n, \delta)$.

[8] For, if x_0 is in the cut locus of p, we may fix $\epsilon > 0$ and replace $F(x)$ by $F_\epsilon(x) = \phi(r_\epsilon(x)/(ar_0))R(x)$ where $r_\epsilon(x) = d(x, \gamma(\epsilon)) + \epsilon$ and γ is a minimal geodesic from p to x_0. We may then apply the elliptic maximum principle to F_ϵ and send $\epsilon \to 0$. See, e.g., §1.2 of Chapter 10 in [**111**] for a more detailed exposition of Calabi's trick.

Consider first the case that $\lambda \geq 0$ (shrinkers and steadies). Using (2.89) and (2.91), we see that

$$0 \leq \frac{2|F|}{n\phi}\left(F - \frac{n\lambda\phi}{2} + \frac{n(1+\theta)^2}{a^2 r_0^2} + \frac{nC_2}{2a}\right) \leq \frac{2|F|}{n\phi}\left(F + \frac{C_3}{a}\right),$$

for an appropriate constant C_3 depending on n, δ, and r_0. So $F(x_0) \geq -C_3/a$ and (2.87) follows.

Now suppose that $\lambda < 0$ (expanders). In this case, (2.89) and (2.91) give

$$\begin{aligned} 0 &\leq \frac{2|F|}{n\phi}\left(F - \frac{n\lambda\phi}{2} + \frac{n(1+\theta)^2}{a^2 r_0^2} + \frac{nC_2}{2a} + \frac{n\lambda\varphi' r}{4ar_0}\right) \\ &\leq \frac{2|F|}{n\phi}\left(F + \frac{C_3}{a} - \frac{n\lambda}{2}\left(\varphi - \frac{\varphi' r}{2ar_0}\right)\right) \\ &\leq \frac{2|F|}{n\phi}\left(F + \frac{C_3}{a} - \frac{n\lambda}{2} + \frac{n\lambda}{2}\left(1 - \varphi + \frac{\varphi' r}{2ar_0}\right)\right). \end{aligned}$$

However, by our construction of φ, specifically by (2.85), we have

$$1 - \varphi\left(\frac{r}{ar_0}\right) + \frac{r}{2ar_0}\varphi'\left(\frac{r}{ar_0}\right) \geq -\varepsilon(\delta)$$

at x_0, so (2.87) follows in this case as well.

From the lower bound on F, we immediately obtain that

$$R(p) = F_{\delta,a}(p) \geq \begin{cases} -C_2/a & \text{if } \lambda \geq 0, \\ (1+\varepsilon)\frac{\lambda n}{2} - \frac{C_1}{a}\lambda & \text{if } \lambda < 0 \end{cases}$$

on $B_{\delta a r_0}(x)$ for all $0 < \delta < 1/10$ and $a > 1/\delta$. Sending $a \to \infty$ for any arbitrary $0 < \delta < 1/10$ and then sending $\delta \to 0$ completes the proof of the scalar curvature lower bounds in Theorem 2.14.

Next, we prove the characterization of the equality case. If R achieves one of these minimum values at some point, that is, if $R(p) = 0$ when $\lambda \geq 0$ or $R(p) = n\lambda/2$ when $\lambda < 0$, then R must coincide everywhere with this minimum value by the strong maximum principle. But then the equation for $\Delta_X R$ implies $|\mathrm{Ric} - (R/n)g|^2 \equiv 0$, and the claim follows.

Finally, suppose in addition that $\lambda > 0$ and the shrinker is gradient. Then we have that $\nabla^2 f = \frac{1}{2}g > 0$ and $f = |\nabla f|^2 \geq 0$. Hence $\inf_{\mathcal{M}} f = f(o) = 0$, where o is the unique critical point of f (which exists by Theorem 4.3 below). Defining $\rho := 2\sqrt{f}$, we have on $\mathcal{M}^n \setminus \{o\}$ that

$$\nabla^2(\rho^2) = 2g \quad \text{and} \quad |\nabla\rho|^2 = 1. \tag{2.92}$$

It now follows from the proof of Proposition 2.9 that $(\mathcal{M}^n, g)$ is isometric to Euclidean space. This completes the proof of the theorem. □

Regarding the lower bound for the scalar curvature, more generally one may consider a solution to the Ricci flow $(\mathcal{M}^n, g(t))$. Then

$$\frac{\partial R}{\partial t} = \Delta R + 2\,|\mathrm{Ric}|^2 \geq \Delta R + \frac{2}{n} R^2 \geq \Delta R. \tag{2.93}$$

Recall from Definition 1.11 that an ancient solution is a solution to the Ricci flow which exists on an interval of the form $(-\infty, \omega)$. The following result for complete ancient solutions is due to B.-L. Chen; see [**86**] for the proof.

Theorem 2.23. *Any complete ancient solution to the Ricci flow must have nonnegative scalar curvature. If the solution has zero scalar curvature at some point and time, then the solution is Ricci-flat at all earlier times.*

Chen's theorem in particular applies to both shrinking and steady Ricci solitons.

2.8. Completeness of the soliton vector field

The equivalence of Ricci solitons and self-similar solutions to the Ricci flow is a fundamental heuristic principle and one that is at least *morally* true. However, the correspondence established in Proposition 2.2 falls short of realizing a true equivalence between the two concepts since the self-similar solution it produces from a Ricci soliton need only be defined locally. In order to properly leverage this correspondence, we will need to know when the two concepts are really the same. The crucial issue is the *completeness* of the Ricci soliton vector field.

Definition 2.24. A vector field X on a manifold $\mathcal{M}^n$ is said to be **complete** if for all $p \in \mathcal{M}^n$ the maximal integral curve $\sigma(t)$ of X with $\sigma(0) = p$ is defined for all $t \in \mathbb{R}$.

In this section, we will present two criteria which guarantee the completeness of the Ricci soliton vector field which together show that in the situations of greatest interest for singularity analysis, the concepts of Ricci solitons and self-similar solutions are indeed equivalent.

The first criterion is completely elementary.

Theorem 2.25 (Completeness of the soliton field, I)**.** *Suppose $(\mathcal{M}^n, g, X, \lambda)$ is a Ricci soliton for which $(\mathcal{M}^n, g)$ is complete and of bounded Ricci curvature. Then X is complete.*

Proof. Fix any point $p \in \mathcal{M}^n$ and let $\sigma : (A, \Omega) \to \mathcal{M}^n$ be the maximal integral curve of X with $\sigma(0) = p$. The completeness of $(\mathcal{M}^n, g)$ and the local theory of ODEs imply that $-\infty \leq A < 0 < \Omega \leq \infty$ and — given the maximality of σ — that if either $A > -\infty$ or $\Omega < \infty$, then $d(p, \sigma(t)) \to \infty$ as $t \searrow A$ or $t \nearrow \Omega$, respectively.

Using the Ricci soliton equation, we compute that the function $t \mapsto |X|^2(\sigma(t))$ satisfies

$$\frac{d}{dt}|X|^2 = 2\langle \nabla_X X, X \rangle = \lambda |X|^2 - 2\mathrm{Ric}(X, X)$$

for all $t \in (A, \Omega)$. Hence, since the Ricci curvature is bounded, there is a constant C such that

$$-2C|X|^2 \leq \frac{d}{dt}|X|^2 \leq 2C|X|^2$$

along σ, and thus

$$\mathrm{e}^{-Ct}|X|(0) \leq |X|(\sigma(t)) \leq \mathrm{e}^{Ct}|X|(\sigma(0))$$

for all $t \in (A, \Omega)$.

From this we see that, if $\Omega < \infty$, then $|X|(\sigma(t)) \leq C'$ for all $t \in [0, \Omega)$. But then, along any sequence $0 \leq t_i \nearrow \Omega$, we would have

$$d(p, \sigma(t_i)) \leq \mathrm{L}(\sigma|_{[0,t_i]}) = \int_0^{t_i} |X|(\sigma(t))\, dt \leq C'\Omega,$$

contradicting the maximality of σ; here, L denotes the Riemannian length. Thus we must have $\Omega = \infty$. A similar argument shows that $A = -\infty$ and hence that $\sigma(t)$ is defined for all $t \in \mathbb{R}$. It follows that X is complete. □

Remark 2.26. Since Theorem 2.14 implies that the scalar curvature of a complete Ricci soliton is bounded below, the two-sided bound on the Ricci curvature in the theorem above may be replaced with merely an upper bound.

The assumption that $(\mathcal{M}^n, g)$ be complete in Theorem 2.25 is certainly necessary: If $(\mathcal{M}^n, g, X, \Lambda)$ is a complete Ricci soliton with a nontrivial (i.e., not identically zero) vector field and $p \in \mathcal{M}^n$ is such that $X(p) \neq 0$, then the restriction of X to $\mathcal{M}^n \setminus \{p\}$ will not be complete. However, the necessity of the assumption of bounded Ricci curvature is less clear. The following result of Z.-H. Zhang [**303**] shows that, at least for *gradient* Ricci solitons, the completeness of the manifold alone is enough to ensure the completeness of the vector field.

Theorem 2.27 (Completeness of the soliton field, II)**.** *Suppose* $(\mathcal{M}^n, g, f, \lambda)$ *is a gradient Ricci soliton for which* $(\mathcal{M}^n, g)$ *is complete. Then* ∇f *is a complete vector field.*

The key to the proof is Hamilton's identity (2.44) and the universal lower bound for scalar curvature proven in Theorem 2.14.

Proof of Theorem 2.27. By combining Theorem 2.14 and (2.44), we have

$$|\nabla f|^2 \leq \lambda f + C \tag{2.94}$$

for some $C = C(\lambda, n) \geq 0$. Fix $p \in \mathcal{M}^n$ and let $r(x) = d(x, p)$.

When $\lambda \neq 0$, (2.94) implies that $h = \lambda f + C$ satisfies $h \geq 0$ and $|\nabla h|^2 \leq |\lambda|^2 h$; that is,

$$|\nabla \sqrt{h}| \leq |\lambda|/2.$$

Choosing $q \in \mathcal{M}^n$ and integrating along any minimizing unit speed geodesic $\gamma : [0, r(q)] \to \mathcal{M}^n$, we find

$$\sqrt{h}(q) - \sqrt{h}(p) = \int_0^{r(q)} \left\langle \nabla \sqrt{h}(\gamma(s)), \gamma'(s) \right\rangle ds \leq \int_0^{r(q)} \left| \nabla \sqrt{h} \right| ds \leq \frac{|\lambda|}{2} r(q).$$

Hence there is a constant $C' > 0$ such that

$$|\nabla f|(q) \leq |\lambda| r(q) + C' \tag{2.95}$$

on all of $\mathcal{M}^n$. On the other hand, when $\lambda = 0$, (2.94) says that $|\nabla f| \leq \sqrt{C}$, so, after possibly enlarging C', estimate (2.95) is valid for all λ. The theorem is now a consequence of the following lemma, which says that the vector field X is complete. □

Lemma 2.28. *Let X be a smooth vector field on $\mathcal{M}^n$. If there is a complete metric g on $\mathcal{M}^n$ relative to which $|X|_g(q) \leq C(d(p,q)+1)$ for some constant C and $p \in \mathcal{M}^n$, then X is complete.*

Proof. Suppose g is a complete metric on $\mathcal{M}^n$ relative to which the growth of $|X| = |X|_g$ is no more than linear relative to the distance $r(q) = d(p, q)$ from some fixed $p \in \mathcal{M}^n$. Fix an arbitrary $q_0 \in \mathcal{M}^n$ and let $\sigma : (A, \Omega) \to \mathcal{M}^n$, $-\infty \leq A < 0 < \Omega \leq \infty$, be any maximal integral curve of X with $\sigma(0) = q_0$.

Now, by assumption, there is a constant $C \geq 0$ such that, for any $t \in [0, \Omega)$, we have

$$\begin{aligned} r(\sigma(t)) &\leq r(q_0) + d(q_0, \sigma(t)) \\ &\leq r(q_0) + \int_0^t |X|(\sigma(s))\, ds \\ &\leq r(q_0) + Ct + C \int_0^t r(\sigma(s))\, ds, \end{aligned}$$

and hence by Grönwall's inequality,

$$r(\sigma(t)) \leq \mathrm{e}^{Ct}(r(q_0) + Ct)$$

for all $t < \Omega$. This shows that $\lim_{t \to \Omega} r(\sigma(t)) = \infty$ only if $\Omega = \infty$. The same argument, applied to the integral curve $t \to \sigma(-t)$ of $-X$, shows that $A = -\infty$, and it follows that X is complete. □

2.9. Compact steadies and expanders are Einstein

On closed manifolds, nonshrinking Ricci solitons are trivial. We have the following result of Ivey [**192**].

Theorem 2.29. *Any steady or expanding Ricci soliton on a closed manifold is Einstein; i.e.,* $\mathrm{Ric} = \frac{r}{n} g$, *where* $r = R_{\mathrm{avg}}$.

Proof. Let $(\mathcal{M}^n, g, X, \lambda)$ be a compact Ricci soliton with $\lambda \leq 0$. Integrating the equation $R + \operatorname{div} X = n\lambda/2$, we see that $r = n\lambda/2 \leq 0$. By taking the divergence of the Ricci soliton equation (2.1), we obtain

$$\Delta X + \mathrm{Ric}(X) = 0. \tag{2.96}$$

From the equation

$$\Delta_X R - \lambda R + 2\,|\mathrm{Ric}|^2 = 0 \tag{2.97}$$

we see that

$$\Delta_X (R - r) + 2\left|\mathrm{Ric} - \frac{r}{n}\right|^2 + \frac{2r}{n}(R - r) = 0. \tag{2.98}$$

Since $\mathcal{M}^n$ is compact, R achieves its minimum value $R_{\min}$ at some $x_0 \in \mathcal{M}^n$, and at any such point

$$2\left|\mathrm{Ric} - \frac{r}{n}\right|^2 + \frac{2r}{n}(R - r) \leq 0.$$

Both terms are nonnegative and thus vanish. In particular, $R_{\min} = R(x_0) = r$, so $R(x) = r$ for all $x \in \mathcal{M}^n$. But then every term in (2.98) must vanish identically on $\mathcal{M}^n$, including $|\mathrm{Ric} - (r/n)g|^2$. □

The theorem is also true in the nongradient case; see Exercise 2.30 for a proof.

2.10. Notes and commentary

The mathematical theory of Ricci solitons was first rigorously developed by Hamilton [**175**–**177**, **179**], laying the foundations of the theory and exhibiting its deep connection to Ricci flow singularity analysis. Bryant, Cao, Ivey, and Koiso made important contributions to the early development of this theory. In the physics literature, the Ricci soliton equation first appeared in Friedan [**151**]. A widely cited survey is by Cao [**61**]. Expository accounts include [**111**, Chapter 4], [**101**, Chapter 1], and [**104**, Chapter 27]. See the extensive references therein on Ricci solitons. Additionally, a selection of papers on Riemannian Ricci solitons and Kähler Ricci solitons, not cited elsewhere in this book, are referenced in the notes and commentary sections of Chapters 4 and 3, respectively.

2.11. Exercises

2.11.1. Scalings and pullbacks of solitons.

Exercise 2.1 (Curvature under scaling)**.** Prove the elementary curvature scaling properties: If α is a positive real number, then

$$\operatorname{Rm}(\alpha g) = \alpha \operatorname{Rm}(g), \quad \operatorname{Ric}(\alpha g) = \operatorname{Ric}(g), \quad R(\alpha g) = \alpha^{-1} R(g). \tag{2.99}$$

Exercise 2.2 (Pullback of curvatures)**.** Let ϕ be a local diffeomorphism. Prove that:

(1) $\operatorname{Rm}_{\phi^* g} = \phi^* \operatorname{Rm}_g$.
(2) $\operatorname{Ric}_{\phi^* g} = \phi^* \operatorname{Ric}_g$.
(3) $R_{\phi^* g} = R_g \circ \phi$.

Exercise 2.3 (Pullback of Lie derivative)**.** Prove that if $\phi : \mathcal{N}^n \to \mathcal{M}^n$ is a diffeomorphism, X is a vector field on $\mathcal{M}^n$, and α is a (covariant) tensor on $\mathcal{M}^n$, then

$$\phi^* (\mathcal{L}_X \alpha) = \mathcal{L}_{\phi^* X} (\phi^* \alpha). \tag{2.100}$$

Exercise 2.4 (Lie derivative of the metric)**.** Prove the Lie derivative of the metric identity (2.28). Generalize this to

$$(\mathcal{L}_X g)_{ij} = \nabla_i X_j + \nabla_j X_i. \tag{2.101}$$

Exercise 2.5 (Lie derivative of the volume form)**.** Prove that the Lie derivative of the volume form is given by

$$\mathcal{L}_X d\mu = \operatorname{div}(X)\, d\mu. \tag{2.102}$$

Exercise 2.6 (Diffeomorphism-invariance of solitons)**.** Prove the diffeomorphism-invariance property (2) for Ricci solitons: If $(\mathcal{M}^n, g, X, \lambda)$ satisfies (2.1) and if $\varphi : \mathcal{M}^n \to \mathcal{M}^n$ is a diffeomorphism, then

$$\operatorname{Ric}_{\varphi^* g} + \frac{1}{2} \mathcal{L}_{\varphi^* X} \varphi^* g = \frac{\lambda}{2} \varphi^* g. \tag{2.103}$$

2.11.2. Product solitons.

Exercise 2.7. Let $(\mathcal{M}_i^{n_i}, g_i)$, $i = 1, 2$, be Riemannian manifolds with Levi-Civita connections ∇_i. Show that the Riemannian product $(\mathcal{M}_1^{n_1}, g_1) \times (\mathcal{M}_2^{n_2}, g_2)$ has Levi-Civita connection ∇ given by

$$\nabla_{X_1 + X_2} (Y_1 + Y_2) = (\nabla_1)_{X_1} Y_1 + (\nabla_2)_{X_2} Y_2 \tag{2.104}$$

for $X_i, Y_i \in T\mathcal{M}_i$, $i = 1, 2$.

Exercise 2.8. Denote the Riemann, Ricci, and scalar curvatures of $(\mathcal{M}_i^{n_i}, g_i)$ by Rm_i, Ric_i, and R_i, respectively.

(1) Prove that the Riemann curvature tensor Rm of the Riemannian product $(\mathcal{M}_1^{n_1}, g_1) \times (\mathcal{M}_2^{n_2}, g_2)$ is given by

$$\begin{aligned}\mathrm{Rm}(X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2, W_1 + W_2) \\ = \mathrm{Rm}_1(X_1, Y_1, Z_1, W_1) + \mathrm{Rm}_2(X_2, Y_2, Z_2, W_2).\end{aligned} \tag{2.105}$$

(2) Prove (2.16), that the Ricci tensor Ric of the Riemannian product satisfies $\mathrm{Ric} = \mathrm{Ric}_1 + \mathrm{Ric}_2$; that is,

$$\mathrm{Ric}(X_1 + X_2, Y_1 + Y_2) = \mathrm{Ric}_1(X_1, Y_1) + \mathrm{Ric}_2(X_2, Y_2). \tag{2.106}$$

(3) Prove that the scalar curvature R of the Riemannian product satisfies

$$R(x_1, x_2) = R_1(x_1) + R_2(x_2) \tag{2.107}$$

for $x_1 \in \mathcal{M}_1^{n_1}, x_2 \in \mathcal{M}_2^{n_2}$.

2.11.3. Nongradient Ricci solitons.

Exercise 2.9 (The Topping–Yin expanding soliton [**278**])**.** Prove that the quadruple $(\mathbb{R}^2, g, X, -1)$ in Example 2.4 satisfies the expanding Ricci soliton equation (2.1) with $\lambda = -1$.

Exercise 2.10. Let $(\mathcal{M}^n, g, X, \lambda)$ be a Ricci soliton. Prove (2.96):

$$\Delta X + \mathrm{Ric}(X) = 0.$$

By taking the divergence of the equation above, prove (2.98):

$$\Delta_X (R - r) + 2\left|\mathrm{Ric} - \frac{r}{n}\right|^2 + \frac{2r}{n}(R - r) = 0.$$

Exercise 2.11 (Compact case of R lower bound)**.** Prove Theorem 2.14 in the case where $\mathcal{M}^n$ is compact. Observe how the proof is simpler than in the noncompact case. The parabolic version of this fact is that on a closed manifold, under the Ricci flow the minimum of the scalar curvature is nondecreasing.

2.11.4. Level sets of the potential function.

Exercise 2.12 (Level sets as evolving hypersurfaces)**.** Let $F : \mathcal{M}^n \to \mathbb{R}$ be a smooth function with $\nabla F(x) \neq 0$ for all $x \in \mathcal{M}^n$. Show that each level set $\Sigma_c := \{F = c\}$ is a smooth hypersurface. Define a 1-parameter group of diffeomorphisms $\phi_t : \mathcal{M}^n \to \mathcal{M}^n$ by $\partial_t \phi_t = \frac{\nabla F}{|\nabla F|^2} \circ \phi_t$, where we assume that $(\mathcal{M}^n, g)$ is complete and the vector field on the right-hand side is complete. Prove that $\phi_t(\Sigma_c) = \Sigma_{c+t}$.

Exercise 2.13. Prove that the second fundamental form, defined by (2.35), is symmetric:

$$\mathrm{II}(Y,X) = \mathrm{II}(X,Y) \quad \text{for } X,Y \in T_x\Sigma_c,\ x \in \Sigma_c. \tag{2.108}$$

Hint: We may extend the vectors X, Y to vector fields defined in a neighborhood $\mathcal{U}$ of x in $\mathcal{M}^n$ so that X, Y are tangent to $\Sigma_c \cap \mathcal{U}$. Note that then $[X, Y]$ is tangent to $\Sigma_c \cap \mathcal{U}$.

Exercise 2.14. Prove the **Gauss equation** for a hypersurface $\Sigma \subset \mathcal{M}^n$ with unit normal vector field ν (if you like, you may assume that Σ is a level set, but this doesn't simplify things): For $X, Y, Z, W \in T_x\Sigma$,

$$\begin{aligned}\mathrm{Rm}_{\mathcal{M}}(X,Y,Z,W) = {} & \mathrm{Rm}_{\Sigma}(X,Y,Z,W) \\ & - \mathrm{II}(X,W)\,\mathrm{II}(Y,Z) + \mathrm{II}(X,Z)\,\mathrm{II}(Y,W).\end{aligned} \tag{2.109}$$

Hint: Extend X, Y, Z, W to vector fields defined in a neighborhood of x and tangent to Σ. Use the formula

$$\nabla^{\mathcal{M}}_X Y = \nabla^{\Sigma}_X Y - \mathrm{II}(X,Y)\,\nu. \tag{2.110}$$

Take the tangential component of the defining equation for $\mathrm{Rm}_{\mathcal{M}}$.

Remark 2.30. The interested reader may take the normal component and derive the **Codazzi equation**:

$$(\nabla^{\Sigma}_X \mathrm{II})(Y,Z) - (\nabla^{\Sigma}_Y \mathrm{II})(X,Z) = -\langle \mathrm{Rm}_{\mathcal{M}}(X,Y)Z, \nu\rangle. \tag{2.111}$$

2.11.5. Special solitons.

Exercise 2.15 (Manifolds with trace-free Ricci tensor)**.** Use the contracted second Bianchi identity (1.60) to prove that if $(\mathcal{M}^n, g)$ satisfies $\mathrm{Ric} = \frac{1}{n} R g$ and $n \geq 3$, then R is a constant. In particular, $(\mathcal{M}^n, g)$ is an Einstein manifold.

Exercise 2.16. Suppose that a quadruple $(\mathcal{M}^n, g, f, \lambda)$ satisfies $\nabla^2 f = \frac{\lambda}{2} g$. Prove that, by adding a constant to f if necessary, we have

$$|\nabla f|^2 = \lambda f. \tag{2.112}$$

Exercise 2.17. Hypothesize as in the previous exercise, now assuming that $\lambda = 1$ and $f > 0$. Define $\rho := 2\sqrt{f}$. Show that $|\nabla \rho| = 1$ and $\nabla_{\nabla\rho} \nabla \rho = 0$. Prove that

$$\mathcal{L}_{\nabla \ln \rho}\left(\frac{g}{\rho^2}\right) = -\frac{4}{\rho^2}\, d\ln\rho \otimes d\ln\rho.$$

2.11.6. Properties of solitons.

Exercise 2.18 (Critical points of f and R)**.** Prove that for any GRS with positive Ricci curvature, if x is a critical point of R, then x is a critical point of f. Does this result hold for negative Ricci curvature?

Exercise 2.19 (Steady GRS have bounded R)**.** Prove that the scalar curvature of any steady GRS is uniformly bounded. Prove that for any steady GRS, if $R \geq 0$ (which is proved later), then $|\nabla f|$ is uniformly bounded.

2.11.7. The f-divergence.

Exercise 2.20. Prove the f-contracted second Bianchi identity*:*

$$\operatorname{div}_f\left(\operatorname{Ric} + \nabla^2 f\right) = \frac{1}{2}\nabla R_f, \tag{2.113}$$

where div_f is defined by (2.62). Derive from this that $R_f + \lambda f$ is constant on a gradient Ricci soliton (for a normalized gradient Ricci soliton we have (2.49)).

Exercise 2.21 (f-divergence theorem)**.** Prove that on a compact Riemannian manifold $(\mathcal{M}^n, g)$ with boundary, for any vector field V we have

$$\int_{\mathcal{M}} \operatorname{div}_f(V) \mathrm{e}^{-f} d\mu = \int_{\partial\mathcal{M}} \langle V, \nu\rangle \, \mathrm{e}^{-f} d\sigma, \tag{2.114}$$

where ν denotes the outward unit normal and where $d\sigma$ is the induced volume element of $\partial\mathcal{M}$. A useful special case is when V is a gradient vector field. For example, we obtain

$$\int_{\mathcal{M}} |\nabla f|^2 \, \mathrm{e}^{-f} d\mu = \int_{\mathcal{M}} \Delta f \, e^{-f} d\mu \tag{2.115}$$

on a closed manifold.

2.11.8. Variation of arc length and Laplacian comparison.

Exercise 2.22. Prove the first variation of arc length formula (2.70).
HINT: Define the map $\Gamma(r, v) := \gamma_v(r)$. Use the formula

$$\partial_v |\gamma'(r)|^2 = 2\left\langle \nabla^\Gamma_V \gamma'(r), \gamma'(r)\right\rangle, \tag{2.116}$$

where ∇^Γ denotes the covariant derivative along the map Γ.

Exercise 2.23. Prove the second variation of arc length formula (2.71).
HINT: Calculate

$$\partial_v|_{v=0} \left\langle \frac{\gamma_v'(r)}{|\gamma_v'(r)|}, \nabla^\Gamma_{\partial_r} V \right\rangle,$$

while using the formula

$$\operatorname{Rm}(V, \gamma_v'(r))V = \nabla^\Gamma_{\partial_v}(\nabla^\Gamma_{\partial_r} V) - \nabla^\Gamma_{\partial_r}(\nabla^\Gamma_{\partial_v} V).$$

Exercise 2.24. Denote $r(x) := d(x,p)$. Prove that, in the strong barrier sense,

$$\Delta r(x) \leq \frac{1}{r(x)} - \frac{1}{r(x)^2}\int_0^{r(x)} r^2 \operatorname{Ric}\left(\gamma'(r),\gamma'(r)\right) dr. \tag{2.117}$$

Exercise 2.25. Let $k \in \mathbb{R}$. Choose $\zeta(r) = \frac{\operatorname{sn}_k(r)}{\operatorname{sn}_k(r_x)}$ in the inequality (2.72) for the Laplacian of the distance function, where

$$\operatorname{sn}_k(r) := \begin{cases} \frac{1}{\sqrt{-k}} \sinh\left(r\sqrt{-k}\right) & \text{if } k < 0, \\ r & \text{if } k = 0, \\ \frac{1}{\sqrt{k}} \sin\left(r\sqrt{k}\right) & \text{if } k > 0. \end{cases} \tag{2.118}$$

What upper bound do you obtain for $\Delta r(x)$?

Exercise 2.26. Let $r_0 \leq r(x)/2$. What second variation inequality do you obtain if you replace $\zeta(r)$ in (2.76) by the slightly more general

$$\zeta\left(r\right) = \begin{cases} \frac{r}{r_0} & \text{if } 0 \leq r \leq r_0, \\ 1 & \text{if } r_0 < r \leq r\left(x\right) - r_0, \\ \frac{r(x)-r}{r_0} & \text{if } r\left(x\right) - r_0 < r \leq r\left(x\right)? \end{cases} \tag{2.119}$$

2.11.9. Maximum principles.

Exercise 2.27 (Elliptic maximum principle)**.** Suppose that a function h with compact support on a complete Riemannian manifold $(\mathcal{M}^n, g)$ satisfies

$$\Delta h + V \cdot \nabla h \geq a h^2 + bh, \tag{2.120}$$

where $a \in \mathbb{R}^+$, $b \in \mathbb{R}$, and V is a vector field. What is the best upper bound for h that you can obtain?

Exercise 2.28 (Weak maximum principle)**.** Prove Lemma B.1 on the elliptic weak and strong maximum principles in Appendix B.
HINT: See Theorem 4 on p. 333 of Evan's book [**145**], which implies that part (2) holds locally on a manifold. Use part (2) to prove parts (1) and (3) by contradiction.

Exercise 2.29. Prove that for a shrinking gradient Ricci soliton $(\mathcal{M}^n, g, f)$, at any minimum point o of f we have $f(o) \leq \frac{n}{2}$.
HINT: Apply the elliptic maximum principle (Lemma B.1) to the equation (2.54) for $\Delta_f f$.

Exercise 2.30 (Formulas for Ricci solitons)**.** Prove that for a Ricci soliton $(\mathcal{M}^n, g, X, \lambda)$:

(1) The function $S := R - \frac{n\lambda}{2}$ satisfies

$$\Delta S - \langle X, \nabla S\rangle + 2\left|\operatorname{Ric} - \frac{\lambda}{2} g\right|^2 + \lambda S = 0. \tag{2.121}$$

(2) Prove Theorem 2.29 for Ricci solitons that are not necessarily gradient.
HINT: When $\lambda \leq 0$, deduce that S is constant by applying the strong maximum principle to (2.121).