§ Gradient Ricci solitons

A gradient Ricci soliton (M, g, f, λ) satisfying $Ric(g) + \nabla^2 f = \frac{\lambda}{2} g$ ，where $\lambda \in R$ 。f is called the potential function。

A Ricci soliton structure is (M, g, X, λ) with $Ric(g) + \frac{1}{2} L_X g = \frac{\lambda}{2} g$

If f is a function，$\nabla f = df$ ，in local coordinates，$\nabla_i f := (df)_i = \frac{\partial f}{\partial x^i}$ and $\nabla^i f := (\nabla f)^i = g^{ij} \nabla_j f$ 。

(*) simplifies to $Ric(g) + \nabla^2 f = \frac{\lambda}{2} g$ since $L_{\nabla f} g = 2\nabla^2 f$ ，here ∇^2 denote the Hessian。These are so-called gradient Ricci solitons。

$Ric(g) + \nabla^2 f = \frac{\lambda}{2} g$ and $L_{\nabla f} g = 2\nabla^2 f$ goes to $-2Ric(g) = L_{\nabla fg} - \lambda g \cdots (*)$

The left-side of the equation is the velocity tensor for Hamilton Ricci flow。

(*) is an underdetermined system of PDEs，the analysis of (*) generally uses the techniques from elliptic and parabolic PDE，from the comparison geometry of Ricci curvature and from Ricci flow。

A shrinking soliton (M,g(t)) $0 \le t < T$ is said to be a gradient shrinking soliton if the vector field X is a gradient of a smooth function on M。

Proposition：

(M,g(0)) is a complete Riemannian manifold，a smooth function $f : M \to R$ ，and a constant $\lambda > 0$ such that $-Ric(g(0)) = Hess(f) - \lambda g(0)$

Then there is T>0 and a gradient shrinking soliton (M,g(t)) defined for $0 \le t < T$

Since $L_{\nabla f} g(0) = 2Hess(f)$ ，$X = \nabla f$

Propsition：

(M, g, X, λ) is a Ricci soliton，if K is a Killing vector field then $(M, g, K + X, \lambda)$ is also a Ricci soliton。

Two Ricci soliton structures $(M_i, g_i, X_i, \lambda_i)$ are equivalent if $\lambda_1 = \lambda_2$ and the underling Riemannian manifolds (M_i, g_i) are isometric。

An isometry $\phi : M_1 \to M_2$ need not pull back X_2 to X_1 。But $\phi^* X_2 - X_1$ at least be a Killing vector field(即 $L_{(\phi^* X_2 - X_1)} g_1 = 0$)。

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1. [Ricci soliton equation]