Ricci flow and the Poincare conjecture Gang Tian

A Ricci soliton is a special solution to the Ricci flow，a geometric flow that evolves a Riemannian metric on a manifold。It is a generalization of an Einstein metric and plays a significant role in understanding the behavior of the Ricci flow and the geometry of manifolds。

A Riemannian metric g on a manifold M is called a Ricci soliton if there exists a smooth vector field X on M and a constant $\lambda \in \mathbf{R}$ such that：

$$\mathrm{R}ic(g)+\frac{1}{2}L_X g=\lambda g$$

Ricci solitons are important because：

1. They arise as fixed points (in the space of metrics modulo scalings and diffeomorphisms) or self-similar solutions(自相似解) of the Ricci flow。Ricci solitons and self-similar solitons are two sides of the same coin。

Proposition 2.2 (Canonical form, I). *Let $(\mathcal{M}^n, g_0)$ be a Riemannian manifold.*

(a) *Suppose that $g(t) = c(t)\phi_t^* g_0$ satisfies the Ricci flow on $\mathcal{M}^n \times (\alpha, \omega)$ for some positive smooth function $c : (\alpha, \omega) \to \mathbb{R}$ and smooth family of diffeomorphisms $\{\phi_t\}_{t\in(\alpha,\omega)}$. Then, for each $t \in (\alpha, \omega)$, there is a vector field $X(t)$ and a scalar $\lambda(t)$ such that $(\mathcal{M}^n, g(t), X(t), \lambda(t))$ satisfies the Ricci soliton equation* (2.1).

(b) *Suppose that $(\mathcal{M}^n, g_0, X, \lambda)$ satisfies the Ricci soliton equation* (2.1) *for some smooth vector field X and constant λ. Then, for each $x_0 \in \mathcal{M}^n$, there is a neighborhood U of x_0, an interval (α, ω) containing* 0, *a smooth family $\phi_t : U \to \mathcal{M}^n$ of injective local diffeomorphisms, and a smooth positive function $c : (\alpha, \omega) \to \mathbb{R}$ such that $g(t) = c(t)\phi_t^* g_0$ solves the Ricci flow on $U \times (\alpha, \omega)$ with $g(0) = g_0$.*

[001RicciSolitonEquation p.4]

在數學與物理中，自相似解指的是形狀在演化過程中保持不變的解，僅僅是透過縮放（scaling）、平移（translation）或其他對稱變換來演化。例如，在偏微分方程或幾何流（Geometric Flows）中，自相似解通常表示某個系統以特定方式發展但不改變其基本結構。

定義 若一個幾何流(例如 Ricci flow) $\frac{\partial g}{\partial t} = -2Ric(g)$ 的某個解滿足

$g(t) = \rho(t)\varphi_t^* g(0)$ 則稱 g(t)為自相似解。

其中 $\rho(t)$ 是一個依賴於時間的縮放因子(scaling factor)，φ_t 是一個依賴於時間的座標變換(diffeomorphism)。

這些解在 Ricci 流中就是 Ricci Soliton，因為它們滿足：$Ric(g) + L_V g = \lambda g$

這表明，在 Ricci 流的演化過程中，幾何形狀基本不變，只是以某種方式變換。

2. They provide insights into the singularities and long-time behavior of the Ricci flow。
3. They are used in the study of the topology and geometry of manifolds, particularly in the context of the Poincaré conjecture and Thurston's geometrization conjecture。

分類：

1. 收縮 $\lambda > 0$
2. 靜態 $\lambda = 0$
3. 膨脹 $\lambda < 0$

Examples：

1. Gaussian Soliton：On Euclidean space R^n，the flat metric is a gradient shrinking soliton。$g_{ij} = \delta_{ij}, V = \frac{x}{2}, \lambda = 1$，對應的 Ricci flow 為 $\frac{\partial g}{\partial t} = -2Ric(g)$

   $g_{ij}(t) = (1 - 2\lambda t) g_{ij}(0)$，當 $\lambda > 0$ 時，這個解會隨時間縮小，導致奇點形成。

   Gaussian cylinder(柱狀孤立子)：$M = R \times S^{n-1}$，$g = dx^2 + e^{2x} g_{round}$ 是一膨脹的 Ricci soliton。Ricci tensor 滿足 $Ric(g) + L_V g = \lambda g$

   其中 $V = \frac{\partial}{\partial x}$，對應於沿 x 方向的自相似膨脹。

2. Shrinking round spheres
3. Einstein manifolds

(M,g) witth $Ric(g)=\frac{1}{2}\lambda g$ of constant scalar curvature $\frac{n\lambda}{2}$，we may associate a Ricci soliton structure of the form (M,g,f,λ) with f=constant。

If a Ricci soliton (M,g,X,λ) is Eiestein with constant $\frac{\lambda}{2}$，then

$L_X g=\frac{1}{2}g-Ric(g)=0$ i.e. X is a Killing vector field。

4. Cigar Soliton：A complete， non-compact steady soliton in two dimensions。

   這是二維非緊致流形上的 Steady Ricci Soliton，形狀類似於一根雪茄，因此得名。其度量可以寫成： $g=\frac{dx^2+dy^2}{1+x^2+y^2}$

   它滿足 Ricci Soliton 方程，且流形在 Ricci 流下保持不變，僅在形狀上發生自相似變化。

5. Bryant Soliton：A complete，non-compact gradient steady soliton in three dimensions。
6. Pseudo-Einstein soliton：在一些更一般的情況下，Perelman 在研究三維流形的幾何化時，發現了非平凡的 Ricci Soliton，例如 標準三維黎卡提空間（Ricci soliton metrics on homogeneous spaces），其中：

- Hamilton 的標準溫度球面解（Hamilton’s standard shrinking soliton on S^3 ）
- Perelman 的 cigar soliton（雪茄孤立子）

Ricci solitons are a central topic in geometric analysis and have deep connections to physics，particularly in the study of general relativity and string theory。

§ Ricci soliton

A Ricci soliton is a Ricci flow， $0\le t<T\le\infty$ ，with the property that for each $t\in[0,T)$ there is a diffeomorphism $\varphi_t:M\to M$ and a constant $\sigma(t)$ such

$g(t)=\sigma(t)\varphi_t^*g(0)...(1)$

$\varphi_t : M \to M$ is a time-dependent family of diffeomorphism with $\varphi_0 = id$ and $\sigma(t)$ is a time-dependent scale factor with $\sigma(0) = 1$。

That is to say，in a Ricci soliton all the Riemannian manifold (M,g(t)) are isometric up to a scale factor that is allowed to vary with t。
The soliton is said to be shrinkng if $\sigma'(t) < 0$ for all t。

(1) 兩邊微分後取 t=0

$$\frac{\partial}{\partial t} g(t) = \frac{d\sigma(t)}{dt} \varphi_t^* g(0) + \sigma(t) \frac{\partial}{\partial t} \varphi_t^* g(0)$$

$-2Ric(g(0)) = \sigma'(0) g(0) + L_V \mathrm{g}(0) \cdots (2)$，where $V = \dfrac{d\varphi_t}{dt}$

A Ricci soliton structure is $(\mathrm{M}, g, X, \lambda)$

$Ric(g) + \dfrac{1}{2} L_X g = \dfrac{\lambda}{2} g \cdots (3)$ 與(2)比較得 $\lambda = -\sigma'(0)$

Tracing (3)，we have $R + divX = \dfrac{n\lambda}{2}$，R is the saclar curvature。

$$divX = tr(\nabla X) = \sum_{i=1}^{n} \nabla_i X^i$$

As before，$g(t) = (1 - 2\lambda t) g_0$ is a solution of the Ricci flow，i.e. a Ricci soliton in Eistein manifolds。

If f is a function，$\nabla f = df$，in local coordinates，$\nabla_i f := (df)_i = \dfrac{\partial f}{\partial x^i}$ and

$\nabla^i f := (\nabla f)^i = g^{ij} \nabla_j f$。

(3) simplifies to $Ric(g) + \nabla^2 f = \dfrac{\lambda}{2} g$ since $L_{\nabla f} g = 2\nabla^2 f$，here ∇^2 denote the Hessian。These are so-called **gradient Ricci solitons**。
A shrinking soliton (M,g(t)) $0 \le t < T$ is said to be a gradient shrinking soliton if the vector field X is a gradient of a smooth function on M。

Proposition
(M,g(0)) is a complete Riemannian manifold，a smooth function $f : M \to R$，and a constant $\lambda > 0$ such that $-Ric(g(0)) = Hess(f) - \lambda g(0)$
Then there is T>0 and a gradient shrinking soliton (M,g(t)) defined for $0 \le t < T$

Since $L_{\nabla f} g(0) = 2Hess(f)$，$X = \nabla f$

[Remark]
Lie derivative of a form ω：

$X \in \chi(M)$，$L_X \omega := \lim_{t\to 0}\frac{1}{t}(\varphi_t^* \omega - \omega) = \frac{d}{dt}(\varphi_t^* \omega)|_{t=0}$，Where φ_t is the local flow of X。

The Lie derivative of the metric tensor g：

$(L_V g)_{ij} = V^k g_{ij,k} + V^k{}_{,i} g_{kj} + V^k{}_{,j} g_{ik}$ Or $(L_X g)_{\mu\nu} = X^\rho \partial_\rho g_{\mu\nu} + g_{\rho\nu}\partial_\mu X^\rho + g_{\rho\mu}\partial_\nu X^\rho$

Note that，K is a Killing vector field $\Leftrightarrow L_K g = 0$

(M, g, X, λ) is a Ricci soliton，then $(M, g, K + X, \lambda)$ is also a Ricci soliton。

Lemma

On a Riemannian manifold (M,g)，we have $(L_X g)_{ij} = \nabla_i X_j + \nabla_j X_i$

Where ∇ denote the Levi-Civita connection of the metric g，for any vector field X。

Let ω be the 1-form due to the vector field X，$\omega(Y) = < X, Y >$ then
$L_X g(Y, Z) = ... = (\nabla_Y \omega)(Z) + (\nabla_Z \omega)(Y)$
Let $\sigma'(0) = 2\lambda$ in the result of Lemma 1.7 to write (1) in coordinates as

$-2R_{ij} = 2\lambda g_{ij} + \nabla_i V_j + \nabla_j V_i$

As a special case we can consider the case that V is the gradient vector field of some scalar function on M^n，i.e. $V_i = \nabla_u f$。The equation then becomes

$R_{ij} + \lambda g_{ij} + \nabla_i \nabla_j f = 0$

Such solutions are known as gradient Ricci solitons。
A gradient Ricci soliton is called shrinking if $\lambda < 0$，static if $\lambda = 0$，and expanding if $\lambda > 0$

§ Special and explicitly defined Ricci solitons

1. The Gaussian solitons
2. Shrinking round spheres

   The metrics of constant positive curvature on the sphere S^n are anturally shrinking gradient Ricci solitons，when paired with any constant potential function。

   If g_{S^n} is the round metric of constant sectional curvature equal to one，the rescaled metric $g = 2(n-1)g_{S^n}$ will satisfy [$Ric(g) + \nabla^2 f = \frac{\lambda}{2} g$]with the canonical choice of constant $\lambda = 1$ 。

We call $(S^n, g, \frac{n}{2})$ the shrinking round sphere 。

For $S^n (n > 1)$ of radius r，the metric is given $g = r^2 \overline{g}$ ，where $\overline{g}$ is the metric on the unit sphere。The sectional curvature are all $\frac{1}{r^2}$ 。

Thus for any unit vector v， $Ric(v, v) = \frac{n-1}{r^2}$ by Lemma 1.11 。

Therefore $Ric = \frac{n-1}{r^2} g = (n-1)\overline{g}$

So the Ricci flow equation becomes an ODE

$$\frac{\partial g}{\partial t} = -2Ric(g) \Rightarrow \frac{\partial}{\partial t}(r^2 \overline{g}) = -2(n-1)\overline{g} \Rightarrow \frac{dr^2}{dt} = -2(n-1)$$

$r(t) = \sqrt{R_0^{\ 2} - 2(n-1)t}$ ，The manifold shrinks to a point as $t \to \frac{R_0^{\ 2}}{2(n-1)}$ 。

Similarly，for hyperbolic n-space $H^n (n > 1)$ ，the Ricci flow reduces to the ODE

$\frac{d(r^2)}{dt} = 2(n-1)$ which has the solution $r(t) = \sqrt{R_0^{\ 2} + 2(n-1)t}$

So the solution expands out to infinity 。

Reference [Curve shorting flow]

3. Hamilton cigar soliton

Let $M = R^2$ ， $g_0 = \rho^2 (dx^2 + dy^2)$

The Gauss curvature $K = -\frac{1}{\rho^2} \Delta \ln \rho$ ， $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Then $Ric(g_0) = K g_0$ ，if we set $\rho^2 = \frac{1}{1 + x^2 + y^2}$ ，we will find $K = \frac{2}{1 + x^2 + y^2}$

That is $Ric(g_0) = \frac{2}{1 + x^2 + y^2} g_0$ ，meanwhile，if we define $Y := -2(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y})$

Then $L_Y g_0 = -\frac{4}{1 + x^2 + y^2} g_0$ ，by (1.2.4) $-2Ric(g_0) = L_Y g_0 - 2\lambda g_0$

$\lambda = 0$ ， g_0 is a steady Ricci flow 。

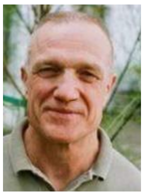
If write g_0 in terms of the geodesic distance from the origin，and polar angle to give

$$g_0 = ds^2 + \tanh^2 s d\theta^2$$

This show that the cigar opens at infinity like a cylinder，and therefore looks like a cigar！

The curvature in these coordinates is $K = \dfrac{2}{\cosh^2 s}$

Finally，note that the cigar is also a gradient soliton since Y is radial。Indeed we may take $f = -2\ln\cosh s$。



4. The Bryant soliton

Robert L. Bryant https://www.msri.org/people/staff/bryant/

[3DXM Surface Gallery]

[The Modeling of Degenerate Neck Pinch Singularities in Ricci Flow by Bryant Solitons]

5. Einstein manifolds $Ric(g) = \dfrac{\lambda}{2} g$ of constant scalar curvature $\dfrac{n\lambda}{2}$

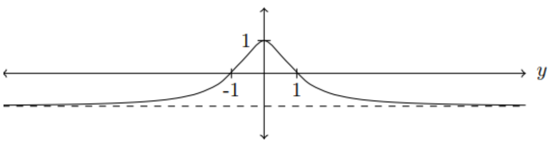
   $$Ric(g) + \frac{1}{2} L_X g = \frac{\lambda}{2} g$$

   If (M, g, X, λ) is Einstein soliton，then $L_X g = 0$

   The vector field X is Killing。
6. Product solitons
7. Quotient solitons
8. Nongradient(無梯度) solitons

   The complete Riemannian metric $g = \dfrac{2}{1+y^2}(dx^2 + dy^2)$，together with the complete vector field $X = -x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}$ generated by homotheties，comprises(包括) a complete nongradient expanding Ricci soliton struture $(R^2, g, X, -1)$ on R^2。

   The scalar curvature of g is given by R(x,y)=$\dfrac{1-y^2}{1+y^2}$



The scalar curvature as a function of y： $R(x,y) = \dfrac{1-y^2}{1+y^2}$

$R_{e^u g_E} = -e^{-u}\Delta u$ with $u = \ln(\dfrac{2}{1+y^2})$，and where Δ is the Euclidean Laplacian。

Reference [hyperbolic plane] $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$

參考資料

1. [Ricci solitons with SO(3)-symmetries] by Robert L. Bryant
2. [Recent progress on Ricci solitons] by Huai-Dong Cao(曹懷東)
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