§ Gaussian cylinder(柱狀孤立子)： $M = R \times S^{n-1}$ ， $g = dx^2 + e^{2x} g_{round}$ 是一膨脹的 Ricci soliton。Ricci tensor 滿足 $Ric(g) + L_V g = \lambda g$

其中 $V = \dfrac{\partial}{\partial x}$ ，對應於沿 x 方向的自相似膨脹。

Ex Find a metric of a Cigar soliton

這是二維非緊致流形上的 Steady Ricci Soliton，形狀類似於一根雪茄，因此得名。其度量可以寫成： $g = \dfrac{dx^2 + dy^2}{1 + x^2 + y^2}$

它滿足 Ricci Soliton 方程，且流形在 Ricci 流下保持不變，僅在形狀上發生自相似變化。

The cigar soliton is a two-dimensional gradient steady Ricci soliton。

It is a complete，rotationally symmetric metric on the plane R^2 。

Here are its common representations： $ds^2 = \dfrac{dr^2 + r^2 d\theta^2}{1 + r^2}$

Where $r \ge 0$ and $\theta \in [0, 2\pi)$

This form shows the metric is conformally equivalent to the Euclidean plane with a conformal factor $\dfrac{1}{1 + r^2}$ 。The Gaussian curvature $K = \dfrac{2}{1 + r^2}$

For the conformal metric $ds^2 = \frac{dr^2 + r^2 d\theta^2}{1 + r^2}$, the Ricci tensor is $Ric = \frac{2}{1 + r^2} g$. The potential $f(r) = -\ln(1 + r^2)$ satisfies $\mathrm{Hess}(f) = -Ric$, confirming the soliton condition.

用直角坐標表示即為 $g = \dfrac{dx^2 + dy^2}{1 + x^2 + y^2}$

The soliton satisfies the steady Ricci soliton equation $Ric + \nabla\nabla f = 0$ with potential $f(x, y) = -\ln(1 + x^2 + y^2)$

Computation in details：

For a conformally flat metric $g_{ij} = \frac{\delta_{ij}}{f(x,y)}$ ，the Christoffel symbols are

$$\Gamma^k_{ij} = \frac{1}{2f}\left(\delta_{ki}\partial_j f + \delta_{kj}\partial_i f - \delta_{ij}\partial_k f\right),$$

Where $f(x,y) = 1 + x^2 + y^2$

$$\Gamma^x_{xx} = \frac{x}{1+x^2+y^2}, \quad \Gamma^x_{xy} = \Gamma^x_{yx} = \frac{y}{1+x^2+y^2}, \quad \Gamma^x_{yy} = -\frac{x}{1+x^2+y^2},$$

and similarly for Γ^y_{ij} by symmetry.

TheGaussian curvatureK for a conformally flat metric $ds^2 = e^{2\phi}(dx^2 + dy^2)$ is given by $K = -\Delta\phi$ ，where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian。

Here $e^{2\phi} = \frac{1}{1+x^2+y^2}$ ，so $\phi = -\frac{1}{2}(1+x^2+y^2)$ ，compute the Laplacian gives $K = \frac{2}{1+x^2+y^2}$

For a 2D metric，the Ricci tensor is Ric=Kg

$$Ric_{ij} = \frac{2}{1+x^2+y^2} \times \frac{\delta_{ij}}{1+x^2+y^2} = \frac{2\delta_{ij}}{(1+x^2+y^2)^2}$$

The potential function is $f(x,y) = -\ln(1+x^2+y^2)$

Compute the Hessian $\nabla\nabla f$ ：

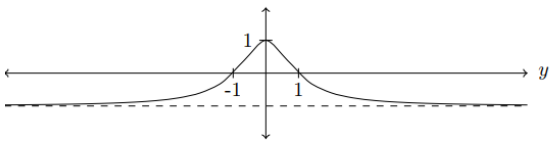
$(\nabla\nabla f)_{ij} = \partial_i\partial_j f - \Gamma^k_{ij}\partial_k f$ ，$\partial_i\partial_j f = -\frac{2}{1+x^2+y^2} + \frac{4x_i x_j}{(1+x^2+y^2)^2}$ where $x_1 = x, x_2 = y$ and simplifying：$(\nabla\nabla f)_{ij} = -\frac{2\delta_{ij}}{(1+x^2+y^2)^2}$

Thus $Ric + \nabla\nabla f = \frac{2\delta_{ij}}{(1+x^2+y^2)^2} - \frac{2\delta_{ij}}{(1+x^2+y^2)^2} = 0$

This confirms the steady Ricci soliton condition。

The complete Riemannian metric $g=\frac{2}{1+y^2}(dx^2+dy^2)$ ，together with the complete vector field $X=-x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}$ generated by homotheties，comprises(包括) a complete nongradient expanding Ricci soliton struture $(R^2,g,X,-1)$ on R^2 。

The scalar curvature of g is given by R(x,y)=$\frac{1-y^2}{1+y^2}$



The scalar curvature as a function of y： $R(x,y)=\frac{1-y^2}{1+y^2}$

$R_{e^u g_E}=-e^{-u}\Delta u$ with $u=\ln(\frac{2}{1+y^2})$ ，and where Δ is the Euclidean Laplacian。

Let $M=R^2$ ， $g_0=\rho^2(dx^2+dy^2)$

The Gauss curvature $K=-\frac{1}{\rho^2}\Delta\ln\rho$ ， $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

Then $Ric(g_0)=Kg_0$ ，if we set $\rho^2=\frac{1}{1+x^2+y^2}$ ，we will find $K=\frac{2}{1+x^2+y^2}$

That is $Ric(g_0)=\frac{2}{1+x^2+y^2}g_0$ ，meanwhile，if we define $Y:=-2(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y})$

Then $L_Y g_0=-\frac{4}{1+x^2+y^2}g_0$ ，by (1.2.4) $-2Ric(g_0)=L_Y g_0-2\lambda g_0$

$\lambda=0$ ， g_0 is a steady Ricci flow。

If write g_0 in terms of the geodesic distance from the origin，and polar angle to give

$g_0=ds^2+\tanh^2 s d\theta^2$

This show that the cigar opens at infinity like a cylinder，and therefore looks like a cigar！

The curvature in these coordinates is $K=\frac{2}{\cosh^2 s}$

Finally，note that the cigar is also a gradient soliton since Y is radial。Indeed we may take $f=-2\ln\cosh s$ 。