§ Ricci flow

§ 01 Preface

The Ricci flow was introduced by Richard Hamilton 1982。[ResearchGate]
[Ricci flow 張樹城]

§ 02 Definittion

We have a Riemannian manifold M with the metric g_0，**the Ricci flow is a PDE** that evolves the metric tensor：$\frac{\partial}{\partial t} g(t) = -2Ric(g(t))$，$g(0) = g_0$

A solution to this equation (or a Ricci flow) is a one-parameter family of metrics g(t)，$(\mathrm{M}, g(\mathrm{t}_0))$ is called the initial condition (or initial metric)。

We hope that the metric will evolve towards one of the Thurston eight fundamental geometric structure。

In **harmonic coordinates** about p，that is to say $\Delta x^i = 0$，we have

$R_{ij} = Ric(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = -\frac{1}{2}\Delta g_{ij} + Q_{ij}(g^{-1}, \partial g)$ where Q_{ij} is a quadratic form in g^{-1} and ∂g

So，the Ricci flow equation $\frac{\partial g}{\partial t} = -2Ric(g) = \Delta g + 2Q_{ij}(g^{-1}, \partial g)$ is a heat equation for the Riemannian metric。(heat equation $u_t = k\Delta u$)

§ 03 Some exact solutions to the Ricci flow

(1) Einstein manifolds

Let g_0 be an Einstein metric：$Ric(g_0) = \lambda g_0$，where λ is a constant。

Then for any constant c>0，setting $g = cg_0$

$$Ric(g) = Ric(cg_0) = Ric(g_0) = \lambda g_0 = \frac{\lambda}{c} g$$

Consider $g(t) = u(t) g_0$ is the solution of the Ricci flow，then

$$\frac{\partial g}{\partial t} = u'(t) g_0 = -2Ric(u(t)g_0) = -2Ric(g_0) = -2\lambda g_0$$

$\therefore u'(t) = -2\lambda, u(t) = 1 - 2\lambda t$，thus $g(t) = (1 - 2\lambda t) g_0$ is a solution of the Ricci flow。

The case $\lambda > 0, \lambda = 0, \lambda < 0$ correspond to shrinking，steady and expanding solutions。

Notice that in the shrinking case the solution exists for $t \in [0, \frac{1}{2\lambda})$ and goes singular at $t = \frac{1}{2\lambda}$ 。

(2) The standard metric on each of S^n, R^n, H^n is Einstein。

(3) CP^n equipped with the Fubini-Study metric，which is induced from the standard metric of S^{2n+1} under the Hopf fibration with the fibers of great circles，is Einstein。

(4) Let h_0 be the round metric on S^2 with constant Gaussian curvature $\frac{1}{2}$ 。

Set $h(t) = (1-t)h_0$ ，then the flow $(S^2, h(t)), -\infty < t < 1$ is a Ricci flow。

We also have the product of this flow with the trivial flow on the line $(S^2 \times R, h(t) \times ds^2), -\infty < t < 1$ 。This is called the standard shrinking round cylinder。

The standard shrinking round cylinder is a model for evolving $\varepsilon - necks$ 。

Definition

Let $(M, g(t))$ be a Ricci flow。An evolving $\varepsilon - neck$ centered at (x, t_0) and defined for rescaled time t_1 is an $\varepsilon - neck$

$\varphi : S^2 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \overset{\cong}{\rightarrow} N \subset (M, g(t))$ centered at (x, t_0) with the property that pull-back

via φ of the family of metric $R(x, t_0) g(t')|_N, -t_1 < t' \leq 0$

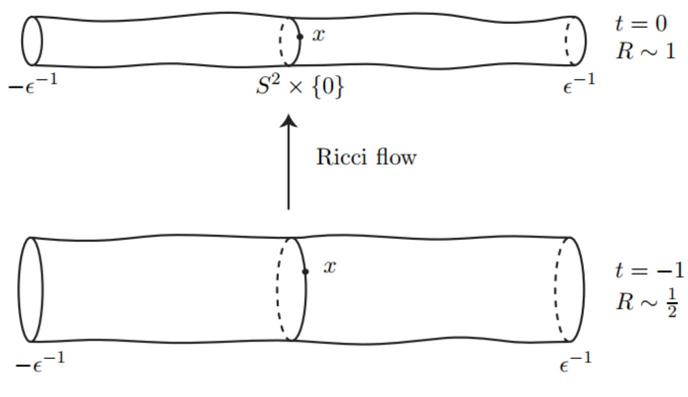

FIGURE 1. Strong ϵ-neck of scale 1.

A strong $\varepsilon - neck$ centered at (x, t_0) in a Ricci flow is an evolving $\varepsilon - neck$ centered at (x, t_0) and defined for rescaled time 1。As left。

§ 04

單位 3 維球 的 metric，$\bar{g} = ds^2 = d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$

半徑 r 的 3 維球，$g = ds^2 = r^2 d\psi^2 + r^2 \sin^2\psi(d\theta^2 + \sin^2\theta d\phi^2)$

此處半徑是時間的函數，$g = r^2 \bar{g}$

n 維球的里奇張量 Ric(g)=(n-1)g，因此 Ricci flow 方程變成常微分方程

$$\frac{\partial g}{\partial t}=-2Ric(g)\Rightarrow\frac{\partial}{\partial t}(r^2\bar{g})=-2(n-1)\bar{g}\Rightarrow\frac{dr^2}{dt}=-2(n-1)$$

$$r^2=R_0^{\ 2}-2(n-1)t$$

$r(t)=\sqrt{R_0^{\ 2}-2(n-1)t}$ ，時間 $t\to\frac{R_0^{\ 2}}{2(n-1)}$ ，此球縮為一點(稱為奇點 singularity)。

Similarly，for hyperbolic n-space $H^n(n>1)$ ，the Ricci flow reduces to the ODE

$\frac{d(r^2)}{dt}=2(n-1)$ which has the solution $r(t)=\sqrt{R_0^{\ 2}+2(n-1)t}$

So the solution expands out to infinity。

§ 05 Singularities in the Ricci flow

§ 06 Short time existence and uniquness of the Ricci flow

§ 07 Ricci solitons

§ 08

The Laplacian：

1. $\Delta u=div(grad(u))$
2. Hessian matrix

$$\mathrm{Hess}(u)=\begin{bmatrix}\frac{\partial^2}{\partial x^2}u & \frac{\partial}{\partial x}\frac{\partial}{\partial y}u & \frac{\partial}{\partial x}\frac{\partial}{\partial z}u\\ \frac{\partial}{\partial y}\frac{\partial}{\partial x}u & \frac{\partial^2}{\partial y^2}u & \frac{\partial}{\partial y}\frac{\partial}{\partial z}u\\ \frac{\partial}{\partial z}\frac{\partial}{\partial x}u & \frac{\partial}{\partial z}\frac{\partial}{\partial y}u & \frac{\partial^2}{\partial z^2}u\end{bmatrix}$$

   is called the Hessian matrix of u，then $\Delta u=trHess(u)$

   the Ricci curvature is the trace of the Riemann curvature tensor。

1. The Ricci flow on the 2-sphere Bennet Chow
2. An Illustrated introduction to the Ricci flow Gabriel Khan [部落格] 內有本書第三版
3. 4D model of the Ricci flow by Dryuma Valery [ResearchGate]
4. Ricci flow gravity by Wolfgang Graf
5. General Relativity and the Ricci flow by Mohammed A. Alzain
6. [Curvature] [Mechanics] [Richard Schoen]