§ Mean Curvature Flow (MCF)

MCF is a process where a surface evolves over time such that each point on the surface moves in the direction of the mean curvature vector。

The surfaces evolve to minimize their area，kind of like how heat equation smooths out temperation distribution。

$\frac{\partial x}{\partial t} = -Hn$ ，where n is the unit normal vector。

It is a parabolic PDE。

MCF is the **negative gradient flow** for the area(volume) functional。

It is a geometric flow that tends to make surfaces more regular over time similar to how harmonic maps minimize energy。

Suppose that M is a closed hypersurface in R^{n+1} and M_t is a variatiom of M。

That is，M_t is a one-parameter family of hypersurfaces with $M_0 = M$ 。

If we think of volume as a finction on the space of hypersurfaces，then the first variation formula gives the derivative of volume under the variation

$\frac{d}{dt} Vol(M_t) = \int_{M_t} \langle \partial_t x, Hn \rangle$ ，here x is the position vector，n the unit normal，and H the mean curvature scalar given by $H = div_M(n) = \sum_{i=1}^{n} \langle \nabla_{e_i} n, e_i \rangle$ where e_i is an orthonormal frame for M。

另一種說法

A geometric diffusion equation $\frac{\partial x}{\partial t} = \Delta_{M_t} x$ for the coordinates x of the corresponding family of surfaces $\{M_t\}_{t \in [0,T)}$ Where ∇_M is the **Laplace-Beltrami operator**。

Since $\Delta_{M_t} x = \overrightarrow{H}$ where $\overrightarrow{H}$ represents the mean curvature vector，we have

$$\frac{\partial}{\partial t} x(p,t) = \overrightarrow{H}(p,t)$$

It follows from the first variation formula that the gradient of volume is $\nabla Vol = Hn$

The most efficient way to reduce the volume is to choose the variation so that

$$\frac{\partial x}{\partial t} = -\nabla Vol = -Hn$$

$\frac{\partial}{\partial t} x = \overrightarrow{H}(x), x \in M_t$ $\overrightarrow{H}(x) = \sum_{i}^{n-1} \lambda_i \nu$ ，λ_i are principal curvatures，ν is the unit normal

Examples

1. For a round sphere，it should shrink homothetially。

$H=\frac{2}{\mathrm{R}}$，$\frac{dR}{dt}=-H=-\frac{2}{R}$

$RdR=-2dt$，$\frac{1}{2}R^2=-2t+C$，the sphere will shrink to a point at $t=\frac{C}{2}$

2. For a cylinder $H=\frac{\frac{1}{R}+0}{2}=\frac{1}{2R}$，Each point collapses at $t={R_0}^2$

3. Planes
4. Torus
5. A dumbbell with a sufficiently long and narrow bar will develop a pinching singularity before extinction。(Grayson)

§ weak solutions of the flow (1)Brakke MCF (2)

§ Shrinkers (homothetic 相似)

§ The shrinker equation

An MCF M_t is a shrinker if and only if $M=M_{-1}$ satisfies the equation $H=\frac{\langle x,n\rangle}{2}$。

That is，$M_t=\sqrt{-t}M_{-1}$ if and only if M_{-1} satisfies $H=\frac{\langle x,n\rangle}{2}$

§ Evolution equation

1. Metric $\frac{\partial}{\partial t}g_{ij}=-2Hh_{ij}$ where h_{ij} is the second fundamental form
2. Area $\frac{\partial}{\partial t}d\mu=-H^2d\mu\rightarrow\frac{d}{dt}Area=-\int H^2d\mu$

   $\frac{d}{dt}Vol(M_t)=-\langle\nabla Vol,\nabla Vol\rangle=-\int_{M_t}H^2$

The simplest case of MCF is when n=1，and the hypersurfaces are curves，this is called curve shortening flow(CSF steepest descent flow for length)。

Theorem (Gage and Hamilton)

Under curve shortening flow，every simple closed convex curve in R^2 remains convex and eventually becomes extinct in a round point。

§ Huisken theorem：[Gerhard Huisken] [ResearchGate]
If the initial surface is uniformly convex，then under MCF，it remains convex and contracts smoothly to a point in finite time，and the rescaled surface converges to a sphere。

§ Maximum principle
1. If two closed hypersurfaces are disjoint，then they remains disjoint under MCF。
2. If the initial hypersurface is embedded，then it remains embedded under MCF。
3. If a closed hypersurface is convex，then it remains convex under MCF。
4. likewise，mean convexity (i.e. $H \geq 0$)is preserved under MCF。

§ Singularities for MCF
§ Applications
1. Image processing
2. Materials science
3. General Ralativity

§ Translating solitons for MCF in R^3
singularities，monotonicity formula，area estimates，comparison principle

§ Documents
1. MCF 大綱 Bulletin f AMS
2. MCF Lecture Notes Brian White Otis Chodosh
3. Singularity of MCF with bounded mean curvature and Morse index Yongheng Han
4. Lectures on MCF and related equations Tom Ilmanen
5. On the topology of translating solitons of the MCF
6. Notes on translating solitons for MCF
   David Hoffman Tom Ilmanen Francisco Martin Brian White
7. Graphical translating solitons for the inverse MCF and iso parametric functions by Tomoki Fujii(藤井朋樹)
8. Any complete immersed two-sided mean convex translating soliton $\Sigma \subset \mathrm{R}^3$ for the MCF is convex。(…bowl soliton)
9. Non-collapsing in mean-convex MCF by Ben Andrews [ResearchGate]
10. Huisken theorem for MCF in sphere Li Lei Hongwei Xu