Brain White
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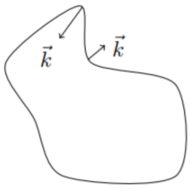
Mean curvature flow is a way to let submanifolds in a manifold evolve。



An embedded curve $\Gamma \subset \mathbb{R}^2$ and its curvature vector $\vec{k}$.

The curvature vector points in a direction which serves to smooth the curve out。

The curve shortening flow is $\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2}$

Where s is the arc-length parameter。

Because s changes with time，this is not the ordinary heat equation，but a non-linear heat equation。However，it still has the nice smoothing properties。

If，for example，Γ is initially C^2，then for t>0 small，Γ_t becomes real analytic。

A ordinary heat equation is $u_t = ku_{xx}, k > 0$

§ Translating soliton
translate 翻譯 轉化
Mean curvature flow：
An example of geometric flow of hypersurfaces in Riemannian manifold。

直觀上，如果曲面上一點移動的速度法向分量由曲面的均曲率給出，則一曲面族在均曲率流下演化。

例如，球面在均曲率流下會透過向內均勻收縮而演化（因為球面的**均曲率向量**指向內）。除特殊情況外，均曲率流都會出現奇點。

在封閉體積恆定的約束下，稱為表面張力(surface tensor)流。

均曲率流在幾何分析，幾何測度理論，偏微分方程，微分拓撲，數學物理…的十字路口。

擴散 diffusion 擾動 perturbation **Laplace-Beltrami operator** denoted as Δ_M
The Laplace-Beltrami operator is the divergence of the (Riemannian) gradient：
$\Delta f = div(\nabla f)$
$L_X dv = (divX)dv$ [Lie derivative]
The divergence of a vector field X on the manifold is defined：

$(\nabla \bullet X)dv := L_X dv$ In local coordinates，one obtains

$\nabla \bullet X = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} X^i)$ then the formular for the Laplace-Beltrami operator aplied to a

scalar function f is，in local coordinates $\Delta f = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} g^{ij} \partial_j f)$

$\frac{\partial u}{\partial t} - \Delta u = 0$ with initial data u_0 and natural boundary condition on $\partial\Omega$

[heat equation]

The geometric diffusion equation $\frac{\partial x}{\partial t} = \Delta_{M_t} x \cdots (*)$ for the coordinates x of the

corresponding family of surfaces $\{M_t\}_{t \in [0,T)}$

A classical formula says that，given a hypersurface in Euclidean space，one has：

$\Delta_{M_t} x = \tilde{H}$，where $\tilde{H}$ represents the mean curvature vector。

This means that (*) can be written as $\frac{\partial}{\partial t} x(p,t) = \tilde{H}(p,t)$

Theorem

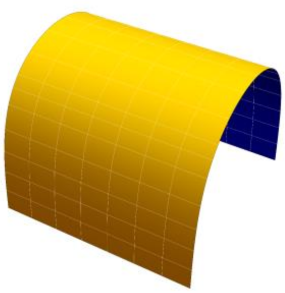
Given a compact，immersed hypersurface M in R^{n+1} then there exists a unique mean curvature flow defined on an interval $[0,T)$ with inatial surface M。

Theorem (Maximum/comparison principle)

If two compact immersed hyperface of R^{n+1} are initially disjoint，they remain so。

Theorem

Convex，embedded，compact hypersurfaces converge to points $p \in R^{n+1}$。After rescaling to keep the area constant，they converge smoothly to round spheres。



Consider the euclidean product $M = \Gamma \times R^{n-1}$

Where Γ is the grim reaper(鐮刀) in R^2 represented by the immersion

$f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to R^2$

$f(x) = (x, \log(\cos x))$

Let M_t be the result of flowing M by mean curvature flow for time t，then

$M_t = M - te_{n+1}$，where $\{e_1, e_2, ..., e_{n+1}\}$ represents the canonical basis of R^{n+1}。In other words，M moves by vertical translations。

§ Translator

A **translator** is a hypersurface M in R^{n+1} such that $t \to M - te_{n+1}$ is a mean curvature flow。i.e. such that normal component of the velocity at each point is equal to the mean curvature at that point ：$\tilde{H} = -e_{n+1}^{\perp}$

The cylinder over a grim-reaper curve，i.e. the hypersurface in R^{n+1} parameterized by $\Upsilon : (-\frac{\pi}{2}, \frac{\pi}{2}) \times R^{n-1} \to R^{n+1}$ given by $\Upsilon(x_1, ..., x_n) = (x_1, ..., x_n, -\log\cos x_1)$ is a translatting soliton。

We can produce others examples of solitons just by scaling and rotating the grim reaper. In this way, we obtain a 1−parameter family of translating solitons parametrized by $\mathscr{G}_\theta : \left(-\frac{\pi}{2\cos(\theta)}, \frac{\pi}{2\cos(\theta)}\right) \times \mathbf{R}^{n-1} \longrightarrow \mathbf{R}^{n+1}$

$$\mathscr{G}_\theta(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \sec^2(\theta)\log\cos(x_1\cos(\theta)) - \tan(\theta)x_n), \qquad (3.2)$$

where $\theta \in [0, \pi/2)$. Notice that the limit of the family F_θ, as θ tends to $\pi/2$, is a hyperplane parallel to $\mathbf{e}_{n+1}$.

§ Variational approach

Tom Ilmanen(1961-)：

A translating soliton M in R^{n+1} can be seen as a minimal surface for the weighted volume functional $A_f[M] = \int_M e^{-f} d\mu$ where f representes the Euclidean height function，that is，the restriction of the last coodinate x_{n+1} to M。

§ Examples

Bowl(碗) soliton (translating paraboloid)

Translating catenoids

Fig. 5. The bowl soliton in $\mathbf{R}^3$ and the translating catenoid for $\lambda = 2$.

§ Graphical translator

If a tranlator M is the grapf of function $u:\Omega \subset \mathrm{R}^n \to \mathrm{R}$ ，we will say that M is a translating graph。

Translator equation $D_i\left(\frac{D_i u}{\sqrt{1+|D_u|^2}}\right) = -\frac{1}{\sqrt{1+|D_u|^2}}$

§ The Spruck-Xiao convexity theorem

§ Omori-Yau theorem

§ Characterization of translting graphs in R^3

參考資料

1. Soliton
2. The evolution of hypersurfaces in Riemannian and Lorentzian manifolds