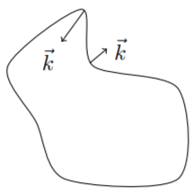
1.1 Curve shortening flow(CSF)

Given a smooth immersion $X_0 : R/Z \to R^2$

We can evolve it by taking $X : R/Z \times [0,T) \to R^2$ such that

$$\begin{cases} \frac{\partial X}{\partial t}(u,t) = \kappa N(u,t) \\ X(u,0) = X_0(u) \end{cases} \cdots \text{This is the CSF equation。}$$

Here κ is the curvature of our curve，N is the inward-pointing unit normal vector，defined by $-\kappa N = \frac{\partial^2 X}{\partial s^2}$



An embedded curve $\Gamma \subset \mathbb{R}^2$ and its curvature vector $\vec{k}$.

The curvature vector points in a direction which serves to smooth the curve out。

The CSF equation is $\frac{\partial X}{\partial t} = \frac{\partial^2 X}{\partial s^2}$。

Where s is the arc-length parameter。

Because s changes with time，this is not the ordinary heat equation，but **a non-linear heat equation**。However，it still has the nice smoothing properties。

If，for example，Γ is initially C^2，then for t>0 small，Γ_t becomes real analytic。

Example：The shrinking circle

If our initial curve X_0 is a circle of radius r_0 centred at the origin，then the solution will take the form $X(u,t) = r(t)(\cos u, \sin u)$。

We have $N = \frac{X}{r}, \kappa = \frac{1}{r}$，so the curve-shortening equation becomes $\frac{dr}{dt} = -\frac{1}{r}$ which has the solution $X(u,t) = \sqrt{r_0^2 - 2t}(\cos u, \sin u)$

So the circle shrinks to a point at a finite tme $t = \frac{r_0^2}{2}$

根據迴避原理(avoidance principle)，包在圓內的曲線會在圓消失之前消失。

§ The acoidance principle

If X,Y are solution of the CSF on $[0,T)$，and $X(u,0) \neq Y(v,0)$ for all u,v，then $X(u,t) \neq Y(v,t)$ for all u,v，for $t \in [0,T)$

That is，if the curves do not intersect initally，they will not intersect at any later time。

Theorem：

If Γ_t is an embedded curve flowing by curve shortening flow，A(t) is the area enclosed by Γ_t，then $A'(t) = -2\pi$

假設Γ_t圍住的區域為Ω，則$A(t) = \iint_{\Omega} dA$

The area swept by the segment ds in time dt is $\approx v_n dtds$，where $v_n = v \cdot n$，here n is the outward-pointing unit normal vector，$v = \frac{\partial X}{\partial t} = \kappa N = -\kappa n$

Then $v_n = v \cdot n = \kappa N \cdot n = -\kappa$

$$dA = \int_{\Gamma_t} v_n dsdt \Rightarrow \frac{dA}{dt} = \int_{\Gamma_t} v_n ds = -\int_{\Gamma_t} \kappa ds = -2\pi$$

$\int_C d\theta = \int_C \frac{d\theta}{ds} ds = \int_C \kappa ds = 2\pi$ 這是切線轉角定理。

1. CSF(或稱 curvature flow) $\frac{\partial X}{\partial t} = \kappa N$ N 朝內
2. A circle $\xrightarrow{t = r^2/2}$ collapse to a point
3. Collision free
4. $A'(t) = -2\pi$
5. Gage-Hamilton theorem 1986

   Convex curves shrink to round points。
6. Grayson theorem 1989

   Under the CSF，embedded curves become convex and thus (by G-H theorem) eventually shrink to round points。