§ Mean Curvature Flow (MCF)

The mean curvature at $p \in S$ is the average of the signed curvature over all angles θ :

$H = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\theta)d\theta = \frac{1}{2}(\kappa_1 + \kappa_2)$ by Euler theorem

MCF is a process where a surface evolves over time such that each point on the surface moves in the direction of the mean curvature vector。

The surfaces evolve to minimize their area，kind of like how heat equation smooths out temperation distribution。

$\frac{\partial x}{\partial t} = -Hn$ ，where n is the unit normal vector。

It is a parabolic PDE。

MCF is the negative gradient flow for the area(volume) functional。

It is a geometric flow that tends to make surfaces more regular over time similar to how harmonic maps minimize energy。

這類流動常見於研究極小曲面、泡沫結構，以及幾何變分問題。

MCF 讓曲面沿著平均曲率方向收縮，最終可能形成特異點。

CMC Flow 則要求演化的曲面始終保持固定的平均曲率，這與肥皂泡的形狀演化有關。

對於嵌入到 R^{n+1} 中的超曲面 M^n ，它的平均曲率 h 定義為主曲率 κ_i 之和的平均值： $H = \frac{1}{n}\sum_i \kappa_i$

CMC flow $\frac{\partial X}{\partial t} = (H - H_0)N$ $X : M^n \to R^{n+1}$ 是曲面的參數化。

當 $H_0 = 0$ CMC flow 就變成標準的 MCF。

應用與研究：

(1) 肥皂泡的形狀：肥皂泡膜在平衡狀態下滿足 CMC 條件，因為表面張力作用使得平均曲率固定。
(2) 廣義相對論：在廣義相對論中，CMC 超曲面可用來研究時空中的時空切片。
(3) 幾何變分問題：研究 CMC 曲面的存在性、穩定性與分類，是微分幾何中的重要課題。

§

均曲率流在蛋白質表面形變理論中的應用，主要涉及生物分子的形狀演化、動力學模擬以及相關的幾何分析。以下是幾個主要的應用方向：

1. 蛋白質摺疊與構象變化

蛋白質在水溶液中的形狀變化（如摺疊與展開）可以視為一種幾何流動過程。在某些模型中，蛋白質的表面可以看作流形，而其變形則可以用均曲率流來描述。均曲率流會驅使高曲率區域趨於平滑，這與蛋白質表面在溶劑影響下的演化過程類似。

2. 生物大分子的自由能最小化

蛋白質表面的形變與自由能密切相關，而均曲率流是一種可以驅使曲面向最小曲面演化的機制。這與某些自由能最小化過程（如基於 Poisson-Boltzmann 方程的電荷調控）相吻合，因此在建構蛋白質溶劑界面的數值方法時，MCF 可用來模擬溶劑可及表面（Solvent Accessible Surface, SAS）或溶劑排除表面（Solvent Excluded Surface, SES）的演化。

3. 蛋白-蛋白與蛋白-配體相互作用

均曲率流在模擬蛋白質表面適應性變形方面也有所應用。當兩個蛋白質相互作用時，界面形狀會發生變化，而這種變化通常受到曲率驅動。MCF 可用於建構蛋白-蛋白或蛋白-配體結合界面形狀的理論模型，幫助理解結合部位的形變機制。

4. 膜蛋白與脂質雙層相互作用

膜蛋白的嵌入與運動涉及蛋白質表面與脂質雙層的形狀適應問題。均曲率流可用來模擬脂質雙層如何根據蛋白質的形狀變化來重新排列，以達到穩定的狀態。例如，在某些膜融合與囊泡形成過程中，均曲率流可用於模擬曲面演化，從而幫助理解生物膜的形態學變化。

5. 數值模擬與計算方法

在計算生物學中，均曲率流常用於發展新的數值演算法，以提高蛋白質表面形變的計算效率。例如，基於水平集方法（Level Set Method）或相場方法（Phase Field Method）的均曲率流數值解法，可用於模擬蛋白質表面形狀的演化，並與分子動力學（Molecular Dynamics）或布朗動力學（Brownian Dynamics）聯合應用。

6. 總結

均曲率流在蛋白質表面形變理論中的應用，主要涉及形狀演化、自由能最小化、相互作用建模以及數值模擬等方面。這一方法提供了一種幾何視角來分析蛋白質的結構變化，並在計算生物學與生物物理學領域具有廣泛的應用價值。

Suppose that M is a closed hypersurface in R^{n+1} and M_t is a variatiom of M。

That is，M_t is a one-parameter family of hypersurfaces with $M_0 = M$。

If we think of volume as a finction on the space of hypersurfaces，then the first variation formula gives the derivative of volume under the variation

$\frac{d}{dt}Vol(M_t) = \int_{M_t} \langle \partial_t x, Hn \rangle$，here x is the position vector，n the unit normal，and H the mean curvature scalar given by $H = div_M(n) = \sum_{i=1}^{n} \langle \nabla_{e_i} n, e_i \rangle$ where e_i is an orthonormal frame for M。

另一種說法

A geometric diffusion equation $\frac{\partial x}{\partial t} = \Delta_{M_t} x$ for the coordinates x of the corresponding family of surfaces $\{M_t\}_{t\in[0,T)}$ Where ∇_M is the Laplace-Beltrami operator。

Since $\Delta_{M_t} x = \vec{H}$ where $\vec{H}$ represents the mean curvature vector，we have

$$\frac{\partial}{\partial t} x(p,t) = \vec{H}(p,t)$$

It follows from the first variation formula that the gradient of volume is $\nabla Vol = Hn$

The most efficient way to reduce the volume is to choose the variation so that

$$\frac{\partial x}{\partial t} = -\nabla Vol = -Hn$$

$\frac{\partial}{\partial t} x = \vec{H}(x), x \in M_t$ $\quad \vec{H}(x) = \sum_{i}^{n-1} \lambda_i \nu$，$\lambda_i$ are principal curvatures，ν is the unit normal

Examples

1. For a round sphere，it should shrink homothetially。

$H = \frac{2}{\mathrm{R}}$，$\frac{dR}{dt} = -H = -\frac{2}{R}$

$RdR = -2dt$，$\frac{1}{2}R^2 = -2t + C$，the sphere will shrink to a point at $t = \frac{C}{2}$

2. For a cylinder $H = \frac{\frac{1}{R} + 0}{2} = \frac{1}{2R}$，Each point collapses at $t = R_0^2$

3. Planes

4. Torus
5. A dumbbell with a sufficiently long and narrow bar will develop a pinching singularity before extinction。(Grayson)

§ weak solutions of the flow (1)Brakke MCF (2)

§ Shrinkers (homothetic 相似)

§ The shrinker equation

An MCF M_t is a shrinker if and only if $M = M_{-1}$ satisfies the equation $H = \frac{\langle x, n\rangle}{2}$。

That is，$M_t = \sqrt{-t}M_{-1}$ if and only if M_{-1} satisfies $H = \frac{\langle x, n\rangle}{2}$

§ Evolution equation

1. Metric $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$ where h_{ij} is the second fundamental form
2. Area $\frac{\partial}{\partial t} d\mu = -H^2 d\mu \to \frac{d}{dt} Area = -\int H^2 d\mu$

   $\frac{d}{dt} Vol(M_t) = -\langle \nabla Vol, \nabla Vol\rangle = -\int_{M_t} H^2$

The simplest case of MCF is when n=1，and the hypersurfaces are curves，this is called curve shortening flow(CSF steepest descent flow for length)。

Theorem (Gage and Hamilton)

Under curve shortening flow，every simple closed convex curve in R^2 remains convex and eventually becomes extinct in a round point。

§ Huisken theorem：[Gerhard Huisken] [ResearchGate]

If the initial surface is uniformly convex，then under MCF，it remains convex and contracts smoothly to a point in finite time，and the rescaled surface converges to a sphere。

§ Maximum principle

1. If two closed hypersurfaces are disjoint，then they remains disjoint under MCF。
2. If the initial hypersurface is embedded，then it remains embedded under MCF。
3. If a closed hypersurface is convex，then it remains convex under MCF。
4. likewise，mean convexity (i.e. $H \geq 0$)is preserved under MCF。

§ Singularities for MCF

§ Applications

1. Image processing
2. Materials science
3. General Ralativity

§ Translating solitons for MCF in R^3

singularities，monotonicity formula，area estimates，comparison principle

§ Documents

1. MCF 大綱 Bulletin f AMS
2. MCF Lecture Notes Brian White Otis Chodosh
3. Singularity of MCF with bounded mean curvature and Morse index Yongheng Han
4. Lectures on MCF and related equations Tom Ilmanen
5. On the topology of translating solitons of the MCF
6. Notes on translating solitons for MCF
   David Hoffman Tom Ilmanen Francisco Martin Brian White
7. Graphical translating solitons for the inverse MCF and iso parametric functions by Tomoki Fujii(藤井朋樹)
8. Any complete immersed two-sided mean convex translating soliton $\Sigma \subset \mathrm{R}^3$ for the MCF is convex。(…bowl soliton)
9. Non-collapsing in mean-convex MCF by Ben Andrews [ResearchGate]
10. Huisken theorem for MCF in sphere Li Lei Hongwei Xu

Exercise

$z = \sinh x\sqrt{1-(\frac{y}{\cosh x})^2}$，求 mean curvature H=

$\sinh x = \frac{e^x - e^{-x}}{2}, \cosh x = \frac{e^x + e^{-x}}{2}$，$\cosh^2 x - \sinh^2 x = 1$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}$$

$$X(x,\theta) = (x, \cosh x \sin\theta, \sinh x \cos\theta)$$

$$X_x = (1, \sinh x \sin\theta, \cosh x \cos\theta)$$

$$X_\theta = (0, \cosh x \cos\theta, -\sinh x \sin\theta)$$

$$E = X_x \cdot X_x = 1 + \sinh^2 x \sin^2\theta + \cosh^2 x \cos^2\theta$$

$$F = X_x \cdot X_\theta = 0$$

$$G = X_\theta \cdot X_\theta = \cosh^2 x \cos^2\theta + \sinh^2 x \sin^2\theta$$

$$X_x \times X_\theta = (-\sinh^2 x - \cos^2\theta, \sinh x \sin\theta, \cosh x \cos\theta)$$

$$\left|X_x \times X_\theta\right|^2 =$$

$$N = \frac{X_x \times X_\theta}{\left|X_x \times X_\theta\right|} = \frac{\cdots}{\sqrt{(\sinh^2 x + \cos^2\theta)(\sin^2 x + \cos^2\theta + 1)}}$$

$$X_{xx} = (0, \cosh x \sin\theta, \sinh x \cos\theta)$$
$$X_{x\theta} = (0, \sinh x \cos\theta, -\cosh x \sin\theta)$$
$$X_{\theta\theta} = (0, -\cosh x \sin\theta, -\sinh x \cos\theta)$$

$$e = \frac{\cosh x \sinh x}{\sqrt{(\sinh^2 x + \cos^2\theta)(\sinh^2 x + \cos^2\theta + 1)}}$$

$$g = \frac{-\cosh x \sinh x}{\sqrt{(\sinh^2 x + \cos^2\theta)(\sinh^2 x + \cos^2\theta + 1)}}$$

$$H = \frac{eG + gE}{2EG} = \frac{e(\text{G-E})}{2EG} = \frac{-e}{2\text{EG}}$$

For fixed x， $(\frac{y}{\cosh x})^2 + (\frac{z}{\sinh x})^2 = \cos^2\theta + \sin^2\theta = 1$

This symmetry implies the surface is **rotationally invariant** around the x-axis，a hallmark(標誌) of minimal surfaces like the catenoid。

The term e contains coshxsinhx in the numerator，but for a minimal surface，**these terms must inherently(本質上) cancel globally** due to the surface's geometric constraints (e.g., the identity $\cosh^2 x - \sinh^2 x = 1$)

Catenoid：

$$\begin{cases} x = c\cosh(\frac{v}{c})\cos u \\ y = c\sinh(\frac{v}{c})\sin u \\ z = v \end{cases}$$

For fixed x， $(\frac{y}{\cosh x})^2 + (\frac{z}{\sinh x})^2 = \cos^2\theta + \sin^2\theta = 1$ is a catenoid。