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The Korteweg-de Vries Equation

Nicolas SCHALCH

schalchn@student.ethz.ch

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Abstract

This report is concerned with a description of the Korteweg-de Vries equation, or KdV equation for short, focusing on special properties of its solutions. Among the latter we have the so called "solitons" which shall be described briefly before giving some historical background on their first observation in 1834 by J.S. Russell. We derive the Korteweg-de Vries equation starting from fluid dynamics and then analyse its additional properties. More precisely, we show that the Korteweg-de Vries equation can be obtained from a 1+1 dimensional field theory from which we find some conserved quantities using the Noether's theorem. Next, the Korteweg-de Vries equation is solved, firstly following the same approach of Korteweg and de Vries in 1895; and secondly by using the inverse scattering transform leading to a larger family of solutions. To conclude, the symmetries of the Korteweg-de Vries equation are analysed and their relation to previously obtained properties of the solutions are discussed.

1 Introduction and Historical Aspects

The concept of solitons is intimately related to the KdV equation. Although a precise mathematical definition of a soliton is hard to establish, we will simply associate solitons to dynamical structures which are solutions to some nonlinear (differential) equations characterized by the following properties [1],[2]:

 (i) Permanent form, i.e. the shape does not change

 (ii) Localized, such that the soliton either decays or approaches a constant at infinity

 (iii) Interact strongly with other solitons and retain its identity after the collision.

Also, we can distinguish between two types of solitons [3]:

- *Topological solitons*: The necessary condition is that there should exist degenerate vacuum states such that the boundary condition at infinity for a soliton is topologically different from that of a physical vacuum state.

- *Non-topological solitons*: The boundary condition at infinity for a non-topological soliton is the same as that for the vacuum state. Thus, there is no need of degenerate vacuum states. The necessary condition for the existence of non-topological solitons is the existence of an additional conservation law.

The very first observation of one single soliton (usually referred to as solitary wave) was made in 1834 by the Scottish engineer John Scott Russell while working at the Union Canal connecting Edinburgh with Glasgow. He was so impressed by the extraordinary constant shape and speed of the wave, that he followed its propagation for a few miles through the canal [4], [5].

FIGURE 1. Illustrates a recreated solitary wave in the Union Canal (1995) [6]-slightly modified.

He described his astonishment with the following words [5]:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation".

With this enthusiasm he started to do some research on these solitary waves during roughly the next decade. His studies were met by scepticism from the scientific community at that time as people were struggling to describe his observations. The reason for that was due to the fact that his observations were a consequence of non-linear effects which were thought to be of secondary importance back then. Despite the critical attitude of his peers he pursued his research and he deicided to stop his experimentations only when two famous mathematicians, namely G.B. Airy and G.G. Stokes, pointed out that his observations were in contradiction with their theories of shallow water waves[1]. It was only in 1877 when the French scientist Joseph Valentine de Boussinesq published a paper including some non-linear terms in his wave equation whose solution exhibited properties similar to the solitary wave originally observed by Russell. However, his paper did not draw much attention in England. Five years later Rayleigh confirmed Russell's observations independently from Boussinesq by obtaining (also) a solitary wave profile. In 1895 Diederik Johannes Korteweg and Gustav de Vries pursued the work done by Rayleigh by including the effect of surface tension leading to the now famous Koerteweg-de Vries (KdV) equation [4], [5], [9]:

$$\frac{\partial \eta}{\partial \tau} + \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^2}\right) = 0. \tag{1.1}$$

In Eq. (1.1) η is the surface elevation (of the wave) above the equilibrium level h, α a constant related to the uniform motion of the liquid (with unit of length), g is the gravitational acceleration and $\sigma := \frac{1}{3}h^3 - \frac{Th}{\varrho g}$ with T denoting the surface tension and ϱ the (fluid) density. A physical example where solitons can be observed (in shallow water) is for instance the tidal bore called "Mascaret" in France. A tidal bore is a wave (or a series of waves) generated by the tide that propagates upstream (of a river). The elevation of the wave and its velocity can reach 5m and 100 km/h respectively. Another example of solitons occurring in nature are tsunamis. Even though oceans are very deep (approximately $h = 4$ km on average) the shallow water theory, more precisely the KdV equation, may still be applied since a tsunami wave can reach a (spacial) extent roughly of 100 km. The propagation speed is calculated using $c_0 = \sqrt{gh} \approx 700$km/h which is very fast. As soon as they approach the coast the depth and the propagation velocity (of the wave) decreases, hence one might intuitively argue that due to the conservation of energy a large wave must rise. The permanent profile of a soliton solution of the KdV equation results from the equilibrium between two effects: non-linearity (proportional to $\eta_\xi\eta$) and dispersion (proportional to $\eta_{\xi\xi\xi}$). Non-linearity tends to comprise the wave whereas dispersion spreads it out. If we return to the tsunami example, we can treat the decreasing depth as a perturbation in the sense that the system cannot reach an equilibrium state. Since the dispersive term is proportional to h^3, it decreases whereas the non-linear term, proportional to $\frac{1}{\sqrt{h}}$, increases, leading to a large wave [4], [5].

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[1]The depth of the water is roughly half of the wavelength (or less) [5].

2 Derivation of the KdV Equation

We shall start this section by introducing some terminology. Let the vector $\boldsymbol{u}(\boldsymbol{x}, t) = \big(u(x, y, t),$ $v(x, y, t)\big)^T$ denote the (two-dimensional) flow velocity at an arbitrary point $\boldsymbol{x}$ and time t. If we assume an incompressible and irrotational flow, mathematically speaking div $\boldsymbol{u} = 0$ and rot $\boldsymbol{u} = 0$, we may rewrite the components u and v of the velocity using the velocity potential ϕ and the stream function φ:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \varphi}{\partial y} \quad \text{and} \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \varphi}{\partial x}$$

which are related to the flow velocity $\boldsymbol{u}$ by grad $\phi = \boldsymbol{u}$ respectively $\boldsymbol{u} = \text{rot}(\varphi \hat{\boldsymbol{z}})$ [7].

In order to demonstrate how Korteweg and de Vries derived their equation, we will start from Euler's equation of fluid dynamics and formulate two physical boundary conditions: the free surface condition and the kinematic boundary condition. Secondly, we will perform a Taylor expansion of the velocity components which will be subsequently inserted into the boundary conditions leading to the KdV equation.

Surface Condition

Euler's equation for an inviscid fluid reads

$$\frac{\partial \boldsymbol{u}}{\partial t} + \big(\boldsymbol{u} \cdot \nabla\big)\boldsymbol{u} = -\nabla\Big(\tfrac{p}{\varrho} + \chi\Big) \tag{2.1}$$

where ϱ is the density of the fluid, p the pressure and $\boldsymbol{g} = \text{grad } \chi$. We may rewrite Eq. (2.1) using the identity $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = (\text{rot } \boldsymbol{u}) \wedge \boldsymbol{u} + \nabla\big(\tfrac{1}{2}\boldsymbol{u}^2\big)$ and the velocity potential ϕ:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\boldsymbol{u}^2 + \chi = C(t) + \frac{p_0 - p}{\varrho} \tag{2.2}$$

where we integrated once, thus obtaining an integration constant p_0 which we identify with the atmospheric pressure, $C(t)$ is an arbitrary function which only depends on time. As already pointed out Korteweg and de Vries included the effect of surface tension[2] which causes a net upward force per unit area of surface of $T\frac{d^2\eta}{dx^2}\Delta x$. This upward force has to correspond to the difference of pressures, in other words $p_0 - p = T\frac{d^2\eta}{dx^2}$. Therefore we rewrite Eq. (2.2):

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}\boldsymbol{u}^2 + g\eta - C(t) = \frac{T}{\varrho}\frac{\partial^2 \eta}{\partial x^2} \quad \text{if} \quad y = \eta(x, t). \tag{2.3}$$

This is known as the free surface condition (with surface tension) [7], [8].

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[2]
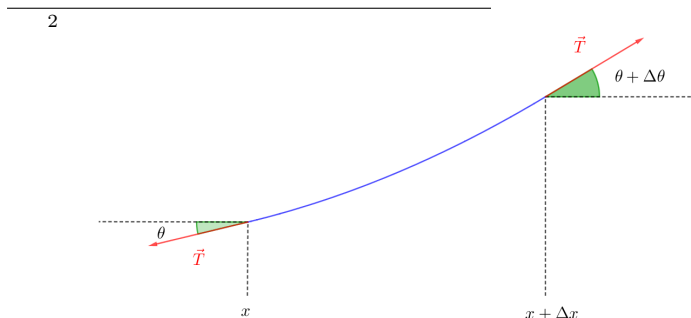
Here we justify the expression for the tension. From the picture on the right we deduce $\Delta T = T\sin(\theta + \Delta\theta) - T\sin(\theta) \approx T\Delta\theta$. On the other hand, differentiate $\tan(\theta) = \frac{dy}{dx}$ leads to $\frac{d\theta}{dx} = \frac{d^2y}{dx^2} \Leftrightarrow \Delta\theta = \frac{d^2y}{dx^2}\Delta x$ where we assumed small angles. As a result, we obtain

$$\Delta T = T \cdot \frac{d^2y}{dx^2} \Delta x$$

Kinematic Condition

Let $\eta(x,t)$ describe the shape of a (one-dimensional) wave[3]. We define a function $F(x,y,t) := \eta(x,t) - y$ which vanishes as long as a particle is on the surface. Taking the (total) derivative with respect to time we obtain

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + (\boldsymbol{u} \cdot \nabla)F = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} - v = 0 \quad \text{if} \quad y = \eta(x,t). \tag{2.4}$$

Eq. (2.4) is referred to as the kinematic boundary condition [7].

The Expansion

Following [9] Korteweg and de Vries expanded the velocity components using Taylor series under the assumption of shallow water $h \ll 1$. By doing so they obtained

$$u(x,t) = f(x,t) - \frac{y^2}{2!}f''(x,t) + \frac{y^4}{4!}f^{(4)} - \ldots \quad \text{and} \quad v(x,t) = -yf'(x,t) + \frac{y^3}{3!}f^{(3)}(x,t) - \ldots \tag{2.5}$$

under the assumption of shallow water $h \ll 1$. In addition, they made the ansatz $f(x,t) = q_0 - \frac{g}{q_0}(\eta(x,t) + \alpha + \gamma(x,t))$ where q_0 is an unknown (constant) velocity, α is a small constant describing the uniform motion of the liquid and γ is small compared to η. By inserting Eq. (2.5) into the kinematic condition (2.4) and into the derivative with respect to time of the free surface condition (2.3), we obtain two equations (for h and γ). Combining these we may eliminate $\gamma(x,t)$ and eventually obtain the Korteweg de Vries equation as it was originally presented in the dissertation of de Vries:

$$\frac{\partial \eta}{\partial t} = \frac{3}{2}\frac{g}{q_0}\frac{\partial}{\partial x}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial x^2}\right) \quad \text{with} \quad \sigma := \frac{1}{3}h^3 - \frac{Th}{\varrho g}. \tag{2.6}$$

Equation (2.6) can be rewritten in a moving frame $\xi := x - (\sqrt{gh} - \sqrt{\frac{g}{h}}\alpha)t$ with $q_0 = -\sqrt{gh}$ and $t = \tau$. In this way we are left with

$$\frac{\partial \eta}{\partial \tau} + \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}\left(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^2}\right) = 0 \tag{2.7}$$

where we neglected the added (constant) velocity [9]. Furthermore, Eq. (2.7) becomes dimensionless by introducing the variables $t := \frac{1}{2}\sqrt{\frac{g}{h}}\tau$, $x := \sigma^{-1/3}\xi$, $u := \sigma^{-1/3}(\frac{1}{2}\eta + \frac{1}{3}\alpha)$, and Eq. (2.7) simplifies to

$$u_t(x,t) + 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0. \tag{2.8}$$

The subscripts in Eq. (2.8) denote partial differentiations. This (simplified) form is how the KdV equation usually appears in the literature. There are at least two several reasons one would like to work with a dimensionless equation: on the one hand, it can be used to describe a wider range of phenomena. On the other hand, it allows more control of the magnitude of the parameters.

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[3]For an illustration see Fig. 2 where $\eta(x,t) = u(x,t)$.

3 Additional Properties of the KdV Equation

The KdV equation also arises as the Euler-Lagrange equation from a Lagrangian density $\mathcal{L}$, in other words it arises from a variational principle. Since the KdV equation is a third order partial differential equation, this suggests that an associated Lagrangian density should also depend on $\partial_\nu \partial_\mu \psi(\boldsymbol{r}, t)$ where the indices take the values $0, ..., 3$. We shall be restricting our discussion to Lagrangian densities which only depend on the field $\psi(x^\mu)$ on 3+1 space-time coordinates $x^\mu = (x^0 = t, x^1 = x, x^2 = y, x^3 = z)$. In short, we consider $\mathcal{L} = \mathcal{L}(\psi(x), \partial_\mu \psi(x), \partial_\nu \partial_\mu \psi(x))$ which does not explicitly depend on x. The equations of motion are obtained from Hamilton's principle of least action $\delta S = \delta \int_{(1)}^{(2)} \mathscr{L} dt = 0$ where δ describes the variation, i.e. the variation of the action vanishes along the path of the motion[4]. Consider the vanishing of the variation of the action:

$$\delta S = \int d^4 x \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta(\partial_\mu \psi) + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \psi)} \delta(\partial_\nu \partial_\mu \psi) \right] \tag{3.1}$$

$$= \int d^4 x \left[\frac{\partial \mathcal{L}}{\partial \psi} \delta \psi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \delta \psi + \partial_\nu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \psi)} \right) \delta \psi \right] = 0, \tag{3.2}$$

where we used the divergence theorem (under the assumption that the field ψ vanishes at infinity) and the fact that the field configuration is given at the times $t^{(1)}$ and $t^{(2)}$ leading to a vanishing surface current. Hence, we obtain the field equations as the (generalised) Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) + \partial_\nu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \psi)} \right) = 0. \tag{3.3}$$

This equation is valid as long as Hamilton's principle of least action holds [10],[11]. As an illustration we consider the Lagrangian density for a field $\psi(x^0, x^1)$:

$$\mathcal{L} = \tfrac{1}{2} \left((\psi_{11})^2 - \psi_1 \psi_0 - 2(\psi_1)^3 \right). \tag{3.4}$$

In Eq. (3.4) we denoted $\psi_1 \equiv \frac{\partial \psi}{\partial x}$ respectively $\psi_0 \equiv \frac{\partial \psi}{\partial t}$. Additionally we have assumed that ψ does not depend on x^2 and x^3. By inserting Eq. (3.4) into Eq. (3.3) we obtain the KdV equation (2.8) if we identify $u = \psi_1$ [13]. It is very useful being able to obtain the KdV equation from a Lagrangian density because this opens the door for using the Noether theorem. The latter leads to the description of conservation laws. However, we are only capable of recognizing very few conserved quantities and it turns out that a system described by the KdV equation has infinitely many conserved quantities [14].

Theorem 3.1 (Noether). *Each (continuous) symmetry transformation which leaves the action invariant corresponds to a conservation law for a specific physical quantity. In particular, the infinitesimal (symmetry) transformation $\psi(x) \mapsto \psi'(x) = \psi(x) + \alpha \Delta \psi$ leads to the conserved current*

$$\partial_\mu j^\mu(x) = 0$$
$$j^\mu(x) := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \Delta \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \psi)} \partial_\nu(\Delta \psi) - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \psi)} (\Delta \psi) - \mathcal{J}^\mu.$$

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[4]We will not explicitly mention the boundary condition(s) of the field (at ∞), for simplicity we assume that we may integrate over the whole (3D-)space and that the field decreases sufficiently rapidly at infinity, so that we can perform a partial integration without obtaining boundary terms [10].

Proof. The (infinitesimal) transformation $\psi(x) \mapsto \psi'(x) = \psi(x) + \alpha\Delta\psi$ maps solutions of the Euler-Lagrange equations to solutions if and only if the action remains invariant under this transformation. Thus, it is sufficient to require that the action only changes by a surface term, since its contribution to the Euler-Lagrange equations would vanish. As a result, in order that $\psi(x) \mapsto \psi'(x) = \psi(x) + \alpha\Delta\psi$ defines a symmetry, the Lagrangian density must transform according to

$$\mathcal{L} = \mathcal{L}(\psi, \partial_\mu\psi, \partial_\nu\partial_\mu\psi) \mapsto \mathcal{L} + \alpha\partial_\mu\mathcal{J}^\mu.$$

On the other hand we compute the variation of $\mathcal{L}$ directly:

$$\alpha\Delta\mathcal{L}$$
$$= \frac{\partial\mathcal{L}}{\partial\psi}\Delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu(\Delta\psi) + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\psi)}\partial_\nu\partial_\mu(\Delta\psi)$$
$$= \alpha\Big[\frac{\partial\mathcal{L}}{\partial\psi} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Big) + \partial_\nu\partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\psi)}\Big)\Big]\Delta\psi + \alpha\partial_\mu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\Delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial\mu\psi)}\partial_\nu(\Delta\psi) - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial\mu\psi)}(\Delta\psi)\Big]$$

Using the Euler-Lagrange equation the first term vanishes while the second term equals $\alpha\partial_\mu\mathcal{J}^\mu$ which eventually leads to Noether's theorem [10], [11]. The attentive reader may have noticed that we have only partially proved the statements of theorem 3.1 which will be required for the subsequent discussion of the KdV equation. In order to complete the proof we would have to consider a transformation of space-time $x^\mu \mapsto x'^\mu$ leading to $\psi'(x')$ which is a lot more difficult. A possible way to proceed would be to generalise the approach followed by [12] for a Lagrangian density (also) depending on the second derivative. $\qquad\square$

We now proceed by applying Noether's theorem to the infinitesimal translation $x^\mu \mapsto x^\mu - \varepsilon^\mu$. This leads to the transformation of the field $\psi(x) \mapsto \psi(x+\varepsilon) = \psi(x) + \varepsilon^\mu\partial_\mu\psi(x)$. Thus, we identify $\alpha\Delta\psi = \varepsilon^\nu\partial_\nu\psi(x)$. The Lagrangian density transforms according to:

$$\mathcal{L}(x) \mapsto \mathcal{L}(x) + \varepsilon^\nu\partial_\nu\mathcal{L}(x) = \mathcal{L}(x) + \varepsilon^\nu\partial_\mu(\delta^\mu_{\ \nu}\mathcal{L}(x)). \tag{3.5}$$

Obviously $\mathcal{J}^\mu$ does not vanish and we get the conserved quantity

$$T^\mu_{\ \nu} := \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi - \delta^\mu_{\ \nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\psi)}\partial_\nu(\partial_\nu\psi) - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\nu\partial_\mu\psi)}(\partial_\nu\psi), \tag{3.6}$$

which we identify as the energy-momentum tensor familiar from electrodynamics. The tensor $T^\mu_{\ \nu}$ completely characterises the energy and the momentum of the field ψ [10], [11]. For instance, if we consider the Lagrangian density $\mathcal{L} = \frac{1}{2}\big((\psi_{11})^2 - \psi_1\psi_0 - 2(\psi_1)^3\big)$ and calculate the components $T^0_{\ 0}$ and $T^0_{\ 1}$ we obtain:

$$T^0_{\ 0} = (\psi_1)^3 - \tfrac{1}{2}(\psi_{11})^2 \equiv \mathcal{H} \quad \text{and} \quad T^0_{\ 1} = -\tfrac{1}{2}(\psi_1)^2.$$

They are respectively the energy density associated with the Hamiltonian(-density) and the momentum density, i.e. the momentum is conserved[5]. We may also shift the (wave-)field ψ by a constant since the Lagrangian density (3.4) is invariant under this (symmetry) transformation:

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[5]The "-" sign comes from

$$T^{\mu\nu} = g^{\varepsilon\nu}T^\mu_{\ \varepsilon}$$
$$T^{00} = g^{00}T^0_{\ 0} = T^0_{\ 0}$$
$$T^{01} = g^{11}T^0_{\ 1} = -T^0_{\ 1}$$

$\psi(x) \mapsto \psi'(x) = \psi(x) + \alpha \cdot \text{const.}$; hence, $\mathcal{J} \equiv 0$ is trivial and $\Delta\psi = \text{const.}$ We compute the first component of the conserved current

$$j^0 = \tfrac{1}{2}\varphi_1 \cdot \text{const.},$$

which is identified as the conserved mass-density. In other words the mass is conserved. To summarize, by reformulating the KdV equation as a variational principle and using the (powerful) Noether theorem in order to obtain conserved quantities, we demonstrated that the mass is conserved due to a translation of φ. The fact that the mass is conserved might not seem surprising since we imposed its conservation in the derivation of the KdV equation. The invariance of Eq. (3.4) with respect to space and time translation respectively imply that momentum and energy are conserved.

As previously mentioned, the KdV equation has an infinite amount of conserved quantities. But how do we obtain these, despite the ones we have already seen? In order to show this claim, it is necessary to define (mathematically) what we interpret as a conserved quantity [14].

Definition 3.1 (Conservation law). *For the partial differential equation*

$$\Delta\Big(x, t, u(x,t)\Big) = 0 \tag{3.7}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ are the temporal and spatial variables and $u(x,t) \in \mathbb{R}$ the dependent variable, a conservation law is an equation of the form

$$\mathscr{D}_t T_i + \mathscr{D}_x X_i = 0 \tag{3.8}$$

which is satisfied for all solutions of the Eq. (3.7), where $T_i(x,t)$, the conserved density, and $X_i(x,t)$, the associated flux, are, in general, functions of x, t, u and the partial derivatives of u; $\mathscr{D}_t$ and $\mathscr{D}_x$ denote the total derivative with respect to the subscript variable.

Under the condition $\lim_{|x|\to\infty} X_i = 0$ sufficiently fast we conclude that an integration of Eq. (3.8) over (the whole) space leads to $\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} T_i(x,t)\,\mathrm{d}x = 0$. After performing the integration we get:

$$\int_{-\infty}^{\infty} T_i(x,t)\,\mathrm{d}x = c_i$$

where c_i (which is constant) is the conserved quantity. In order to apply the previously outlined approach to find conservation laws associated to the KdV equation, it is useful to rewrite $u = w - \varepsilon w_x - \varepsilon^2 w^2$ where w is a function (to be determined) and ε a parameter, which is occasionally referred to as the Gardner transformation. By inserting the latter expression for u into the KdV equation $u_t + 6uu_x + u_{xxx} = (1 - \varepsilon\partial_x - 2\varepsilon^2 w)(w_t + 6(w - \varepsilon^2 w)w_x + w_{xxx})$, we find that u solves the KdV equation if and only if w is a solution of

$$(w_t + 6(w - \varepsilon^2 w)w_x + w_{xxx}) = \frac{\partial}{\partial t}(w) + \frac{\partial}{\partial x}(w_{xx} + 3w^2 - 2\varepsilon^2 w^3) = 0. \tag{3.9}$$

Eq. (3.9) was rewritten in the same form of Eq. (3.8) and consequently the quantity $\int_{-\infty}^{\infty} w\,\mathrm{d}x$ is conserved. Since the KdV equation does not depend on the parameter ε, neither will its solution $u(x,t)$. Moreover $w \to u$ as $\varepsilon \to 0$. Hence, we represent w as a (formal) power series $w(x,t;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n w_n(x,t)$ where we would like to emphasize that the conserved quantities are $\int_{-\infty}^{\infty} w_n(x,t)\,\mathrm{d}x \; \forall n$. By substituting the latter series expansion in $u = w - \varepsilon w_x - \varepsilon^2 w^2$ and

by equating recursively powers of ε we obtain:

$$w_0 = u, \tag{3.10}$$

$$w_1 = w_{0,x} = u_x, \tag{3.11}$$

$$w_2 = w_{1,x} + w_0^2 = u_{xx} + u^2, \tag{3.12}$$

$$w_3 = w_{2,x} + 2w_0 w_1 = u_{xxx} + 4uu_x, \tag{3.13}$$

$$w_4 = w_{3,x} + 2w_0 w_2 + w_1^2 = u_{xxxx} + 6uu_x + 5u_x^5 + 2u^3, \tag{3.14}$$

and so on. The coefficients w_n yields an infinite number of conserved densities. However, in order to obtain the conservation laws of the KdV equation it is necessary to insert Eq. (3.10)-(3.14) into Eq (3.9) and equate the corresponding powers of ε. It turns out that only even power of ε give independent conservation laws while odd powers provide no additional information since these are the derivative of the previous even power. Following the explained procedure we obtain the first three conservation laws after some elementary transformations:

$$(u)_t = -\partial_x(3u^2 + u_{xx}), \tag{3.15}$$

$$(u^2)_t = -\partial_x(4u^3 + 2uu_x - u_x^2), \tag{3.16}$$

$$(u^3 - \tfrac{1}{2}u_x^2)_t = -\partial_x\left(\tfrac{9}{2}u^4 + 3u^2 u_{xx} - 6uu_x^2 - u_x u_{xxx} + \tfrac{1}{2}u_{xx}^2\right). \tag{3.17}$$

As a result we obtain the same conserved quantities as before using Noether's theorem [1], [14]. The reason we previously obtained less conservation laws might be due to the fact that we only considered geometrical transformations.

4 Solving the KdV Equation

In this section we shall examine solutions of the KdV equation[6] and the methods one can use for obtaining them. We first start by looking for travelling wave solutions. For that we make the ansatz $u(x,t) = f(x - ct)$ for some $c \in \mathbb{R}$. By inserting this into the KdV equation yields an ordinary differential equation for f, namely $-cf - 6ff' + f''' = 0$, which can be easily integrated yielding $-cf - 3f^2 + f'' = a$ where a is an integration constant. We then multiply the latter equation by f', integrate it again, thus obtaining an additional integration constant b. By rescaling the (integration) constants, we find:

$$(f')^2 = 2f^3 + cf^2 + a'f + b' \equiv \mathscr{F}(f). \tag{4.1}$$

We have thus obtained a first order non-linear (ordinary) differential equation which needs to be solved. Obviously the roots of $\mathscr{F}$ will play an important role in our analysis and we can distinguish the following cases:

1. $\mathscr{F}$ has three distinct (real) roots.

2. $\mathscr{F}$ has two (real) roots, one has order two.

3. $\mathscr{F}$ has one simple root (and two imaginary ones).

4. $\mathscr{F}$ has one root of order three.

The cases three and four will not be treated here. We will start with the second case and assume that the double root β is larger than the simple root α. It can be shown that if a solution f has initial condition $\alpha < f(x_0) < \beta$, then $\alpha < f(x) < \beta$ holds $\forall x$ and one can expect bounded solutions. Hence we write $\mathscr{F}(f) = 2(f - \beta)^2(f - \alpha)$. By inserting the latter expression for $\mathscr{F}$ in Eq. (4.1) and using some algebraic manipulations we are left with

$$\int_\alpha^f \frac{d\xi}{(\xi - \beta)\sqrt{\xi - \alpha}} = \pm\sqrt{2}x + C \tag{4.2}$$

where C is an integration constant. In order to perform the integration we substitute $\xi := \alpha + (\beta - \alpha)\sin^2(\nu)$. The integral becomes $\frac{-2}{\sqrt{\beta - \alpha}} \int_0^\phi \frac{1}{\cos(\nu)} d\nu$. The quantity ϕ is defined by $\sin(\phi) = \sqrt{\frac{f - \alpha}{\beta - \alpha}}$. Using these substitutions one can perform the required integration by elementary means, yielding

$$\frac{-2}{\sqrt{\beta - \alpha}} \log \mid \tan\left(\tfrac{\phi}{2} + \tfrac{\pi}{4}\right) \mid = \pm\sqrt{2}x + C.$$

By using the addition formula $\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$, we obtain $\frac{1 + \tan\left(\frac{\phi}{2}\right)}{1 - \tan\left(\frac{\phi}{2}\right)} = e^y$ and consequently $\tan\left(\frac{\phi}{2}\right) = \tanh(\frac{y}{2})$ with $y := \pm\sqrt{\frac{\beta - \alpha}{2}}(x - C)$. It is now possible to determine f as a function of x by using the identities $\cos\phi = 1/(1 + \tan^2\phi)$, $\frac{1}{2}(1 + \tanh^2(\frac{y}{2}))\tanh(y) = \tanh(\frac{y}{2})$

---

[6]In this section we shall be referring to $u_t(x,t) - 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0$ as the KdV equation where the substitution $u(x,t) \mapsto -u(x,t)$ was performed.

and that cosh is an even function. In this way we find

$$f(x) = \alpha + (\beta - \alpha)\sin^2(\phi) = \alpha + 4(\beta - \alpha)\tan^2(\tfrac{\phi}{2})\cos^4(\tfrac{\phi}{2})$$

$$= \alpha + 4(\beta - \alpha)\frac{\tanh^2(\tfrac{y}{2})}{(1 + \tanh^2(\tfrac{y}{2}))^2} = \beta + \frac{\alpha - \beta}{\cosh^2(y)}$$

$$= \beta + \frac{\alpha - \beta}{\cosh^2\left(\sqrt{\tfrac{\beta - \alpha}{2}}(x - C)\right)}.$$

We then impose the boundary condition $\lim_{|x| \to \infty} f(x) = 0$ which can be achieved only if $\beta = 0$. Besides that, the roots α and β are related to c according to $c = -2\alpha - 4\beta = -2\alpha$. The latter can be seen by equating the f^2 coefficients in the polynomial $\mathscr{F}$. As a result we derived a solution (for $c > 0$) of the KdV equation reading

$$u(x,t) = \frac{-c}{2\cosh^2\left(\tfrac{1}{2}\sqrt{c}(x - ct + C)\right)}. \tag{4.3}$$

This solution was historically very important because its shape corresponds to Russell's observation. We remark here that the maximum of u lies at $x = ct$, leading to an amplitude of $\tfrac{1}{2}c$. Thus the (maximum) amplitude and the propagation speed are related [16]. Figure 2 shows the propagation of such a wave.
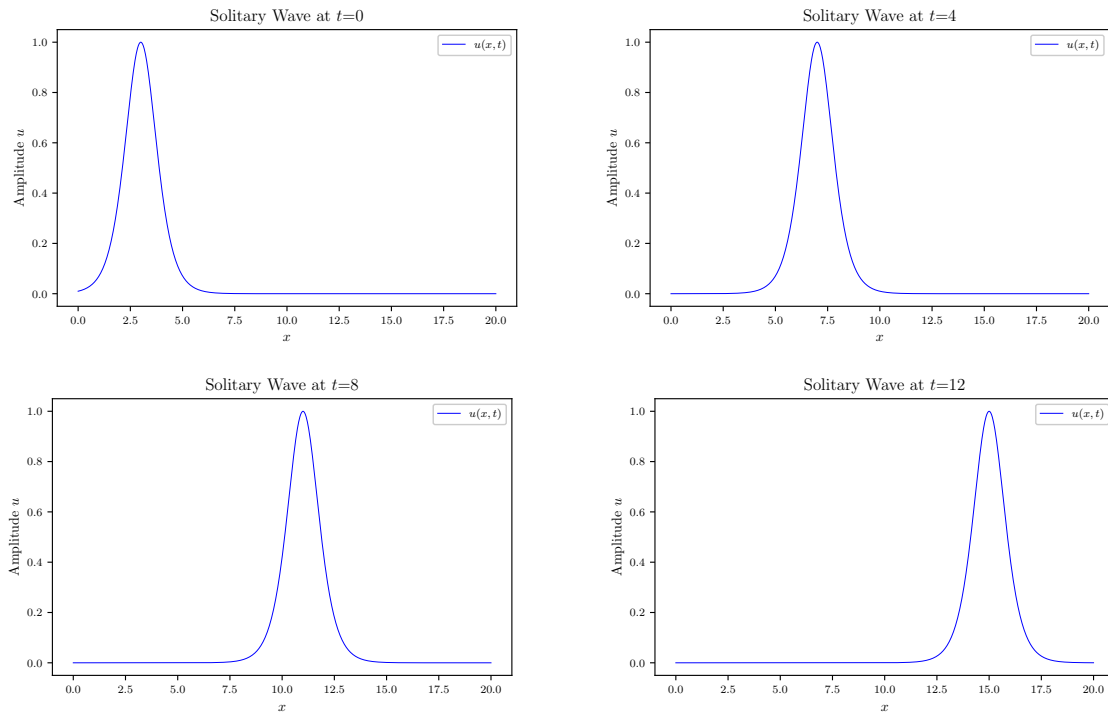

FIGURE 2. Propagation of a solitary wave travelling from the left to the right with $c = 1$.

More general travelling solutions also exist, for instance when $\mathscr{F}$ has three distinct real roots. In order to solve the KdV equation in that case we can proceed in the same way. After performing the substitution $\xi := \alpha + (\beta - \alpha)\sin^2(\nu)$ we then get an integral of the form

$$v := \int_0^\phi \frac{1}{\sqrt{1 - k^2\sin^2\nu}}d\nu. \tag{4.4}$$

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By using the Jacobian elliptic (cosine) function cn we can write the solution as $\cos\phi = \text{cn}(v; k)$. These solutions are referred to as cnoidal waves [16]. Until the sixties these solutions were the only ones known. However, as the KdV equation also appeared in different fields of physics, other solutions than travelling waves were needed in order to describe the observed phenomena. The objective now is to outline the methods needed to solve the KdV equation in more general cases and then apply them to a special family of potentials, namely the reflectionless potentials. In short, we (still) wish to solve the equation

$$u_t(x,t) - 6u(x,t)u_x(x,t) + u_{xxx}(x,t) = 0 \tag{4.5}$$

for a given initial condition $u_0 := u(x, t = 0) = f(x)$ where $f(x)$ decays sufficiently rapidly as $\|x\| \to \infty$. For that Gardner, Greene, Kruskal and Miura proposed the so called Inverse Scattering Method which is based on a relation between the KdV equation (4.5) and the time-independent Schrödinger equation. In order to understand this relation we start by discussing the following proposition:

Proposition 4.1 (Miura)**.** *If v is a solution to the modified KdV equation $v_t - 6v^2 v_x + v_{xxx} = 0$. Then*

$$u = (v^2 + v_x) \tag{4.6}$$

solves the KdV equation.

Proof. This follows from an elementary calculation. We insert $u = (v^2 + v_x)$ into the KdV equation: $u_t - 6uu_x + u_{xxx} = -(2v + \partial_x)(v_t - 6v^2 v_x + v_{xxx}) = 0$ and the proposition follows. $\square$

We recognise Eq. (4.6) as a Ricatti equation which is linearised by making the ansatz $v = \frac{\psi_x}{\psi}$: $\psi_{xx} - u\psi = 0$. Using the Galilean invariance[7] of the KdV equation, i.e. that $u(x,t) \to u(x + 6\lambda t, t) + \lambda$ where $\lambda \in \mathbb{R}$ also satisfies the KdV equation, completes the connection since we obtain

$$\psi_{xx} + \Big(\lambda(t) - u(x,t)\Big)\psi = 0 \tag{4.7}$$

where we allowed the eigenvalue $\lambda = \lambda(t)$ to depend on t.

Since the Schrödinger equation (4.7) is familiar from quantum mechanics, it is more convenient to solve this problem where t is a parameter and $u(x,t)$, the solution we are interested in, the potential. Following this line of reasoning we proceed in the following way in order to solve the initial value problem for the KdV equation (4.5):

1. At time $t = 0$, given $u(x, 0)$ we solve the direct scattering problem and obtain the scattering data $S(\lambda, t = 0)$ at time $t = 0$.

2. By performing the time evolution of the scattering data when the potential evolves according to the KdV equation we then obtain the time evolved scattering data $S(\lambda, t)$.

3. It is now possible to reconstruct the potential $u(x, t)$, which is the sought solution of the KdV equation, from the scattering data at time t. This is usually referred to as the inverse scattering transform. This is essentially done by solving the Gelfand-Levitan-Marchenko (integral) equation which is highly non-trivial.

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[7]This shall be proven in section 5.

The aforementioned steps are depicted in Fig. 3. We would like to emphasize the similarity to the familiar Fourier transform which fails to solve non-linear partial differential equations. In other words, the inverse scattering transform can be seen as a generalisation of the Fourier transform. Since the Gelfand-Levitan-Marchenko equation, the time evolution of the scattering data (which we will see shortly) as well as the Schrödinger equation are linear, we (only) have to solve linear problems in order to obtain a solution of the KdV equation [1], [14], [16]. We will now provide a detailed discussion of the different steps needed to solve the KdV equation.
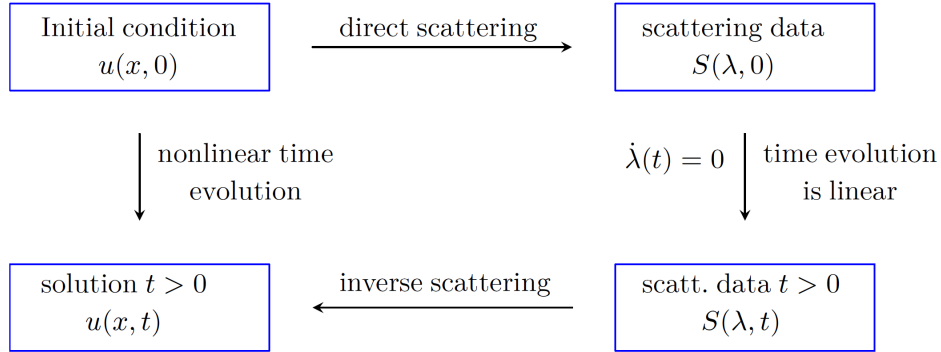

FIGURE 3. Schematic description of the steps involved in the inverse scattering method.

1. **Direct Scattering**

   We will assume that $u(x,0)$ has a compact support, hence we may analyse the asymptotic behaviour of the eigenfunctions which satisfy the Schrödinger equation $-\partial_{xx}\psi + u(x,0)\psi = \lambda\psi$. We consider two different cases: (i) bound states, i.e. discrete (negative) eigenvalues $\lambda = -\kappa_n^2, n = 1, 2, ..., N$ which lead to the eigenfunctions

   $$\psi_n(x,0) = N_n(0)e^{-\kappa_n x}$$

   for sufficiently large x where $N_n(0)$ are the normalisation constants; (ii) a continuum of (positive) eigenvalues $\lambda = k^2$ corresponding to the eigenfunctions

   $$\psi_k(x,0) \sim \begin{cases} T(k,0)e^{-ikx} & \text{as } x \to -\infty \\ e^{-ikx} + R(k,0)e^{ikx} & \text{as } x \to \infty \end{cases}$$

   where $T(k,0)$ and $R(k,0)$ represent the transmission and reflection coefficients.
   We assume that the time dependence is reflected (only) in the scattering data. Finally we obtain the scattering data at time $t = 0$ [14]:

   $$S(\lambda,0) = \left(\{\kappa_n, N_n(0)\}_{n=1}^N, R(k,0), T(k,0) \right).$$

2. **Time Evolution of the Scattering Data**

   In order to describe the time evolution of the scattering data, we rewrite the KdV equation 4.5 as a non-linear evolution equation $u_t = \mathcal{N}(u)$ with $\mathcal{N}$ a non-linear map on a suitable function space defined by $\mathcal{N}(f) := 6ff_x - f_{xxx}$. This allows us to follow a more general procedure which was proposed by Lax in 1967 [1]:

Theorem 4.1 (Lax). *Let M and L be self-adjoint operators, $[L, M] := LM - ML$ denote their commutator and $\lambda \in \mathbb{R}$. If the evolution equation $u_t - \mathcal{N}(u) = 0$ can be expressed as the Lax equation*

$$L_t + LM - ML = L_t + [L, M] = 0 \tag{4.8}$$

and if

$$L\psi = \lambda\psi$$

then $\lambda_t = 0$ and ψ evolves according to

$$\psi_t = M\psi \qquad \text{for} \quad t > 0. \tag{4.9}$$

Proof. We differentiate $L\psi = \lambda\psi$ w.r.t t: $L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t$. By using the Lax equation $L_t + [L, M] = 0$ we can reformulate the latter equation:

$$\lambda_t\psi = (L - \lambda)\psi_t + (ML - LM)\psi = (L - \lambda)\psi_t + M\lambda\psi - LM\psi = (L - \lambda)(\psi_t - M\psi). \tag{4.10}$$

Taking the inner product with $\langle\psi|$ we obtain

$$\langle\psi|\psi\rangle\,\lambda_t = \langle\psi|(L - \lambda)(\psi_t - M\psi)\rangle = \langle(L - \lambda)\psi|\psi_t - M\psi\rangle = 0$$

since L is self-adjoint. Hence, we get that (wlog $\psi \neq 0$) $\lambda_t = 0$, i.e. each eigenvalue of the self-adjoint operator L is constant.

We can rewrite Eq. (4.10) yielding $L(\psi_t - M\psi) = \lambda(\psi_t - M\psi)$, therefore we see that $(\psi_t - M\psi)$ is an eigenfunction of the operator L and we deduce that $\psi_t - M\psi \propto \psi$. Since we are allowed to add a scalar function (multiplied by the identity) to M without violating Lax's equation, we conclude that the time evolution of a state is given by

$$\psi_t = M\psi \quad \text{for} \quad t > 0.$$

$\square$

As soon as we have found a Lax pair L and M which satisfy the Lax equation (4.8), Lax's theorem specifies the time evolution of the states, respectively of the scattering data. Unfortunately, there exists no systematic method to find these associated operators for a given non-linear partial differential equation. However, in the case of the KdV equation the reader may verify that the operators $L := -\frac{d^2}{dx^2} + u(x, t)$ (Schrödinger operator) and $M := -4\frac{d^3}{dx^3} + 6u(x, t)\frac{d}{dx} + 3u_x(x, t)$ do form a Lax pair [16].

Outside the support of $u(x, t)$ M reduces to $M = -4\frac{d^3}{dx^3}$ and the evolution of $\psi_n(x, t)$ is defined by Eq. (4.9)

$$\frac{d}{dt}\psi_n(x, t) = M\psi_n(x, t) = -4\frac{d^3}{dx^3}N_n(t)e^{-\kappa_n x} = 4\kappa_n^3 N_n(t)e^{-\kappa_n x}.$$

On the other hand, $\frac{d}{dt}\psi_n(x, t) = \frac{d}{dt}N_n(t)e^{-\kappa_n x} = e^{-\kappa_n x}\dot{N}_n(t)$ must hold according to our assumption for sufficiently large x. By equating both terms for $\dot{\psi}_n$ we get $\frac{dN_n}{dt}(t) = 4\kappa_n^3 N_n(t)$, hence $N_n(t) = N_n(0)e^{4\kappa_n^3 t}$ which can be reformulated as a (probability) density with $N_n(0) = \sqrt{\rho_n(0)}$ leading to $\rho_n(t) = \rho_n(0)e^{8\kappa_n^3 t}$ [16].

For the time evolution of the reflection and transmission coefficients, we apply a similar procedure using Lax's theorem. We naively assume that the eigenfunction may be written as $\Psi_k(x, t) = A(t)e^{-ikx} + E(t)e^{ikx}$ with $A(0) = 1$, $E(0) = R(k, 0)$ for x large enough. Using again $M = -4\frac{d^3}{dx^3}$ we obtain

$$\dot{\Psi}_k(x, t) = M\Psi_k(x, t) = -4(-ik)^3 A(t)e^{-ikx} - 4(ik)^3 E(t)e^{ikx} = \dot{A}(t)e^{-ikx} + \dot{E}(t)e^{ikx}.$$

This yields $\dot{A} = -4ik^3 A(t)$ and $\dot{E}(t) = 4i\kappa^3 E(t)$ since $e^{\pm ikx}$ are linearly independent. Therefore we obtain the solutions $A(t) = e^{-4ik^3t}$ and $E(t) = R(k,0)e^{4ik^3t}$. Turning to the case where x is largely negative we assume the eigenfunctions $\Psi_k(x,t) = C(t)e^{ikx} + D(t)e^{-ikx}$ with $C(0) = 0$, $D(0) = T(k,0)$. Applying the time evolution operator M we obtain:

$$\dot{\Psi}_k(x,t) = M\Psi_k(x,t) = -4(ik)^3 C(t)e^{ikx} - 4(-ik)^3 D(t)e^{-ikx} = \dot{C}(t)e^{ikx} + \dot{D}(t)e^{-ikx}.$$

This yields $\dot{C} = 4ik^3 C(t)$ and $\dot{D}(t) = -4i\kappa^3 D(t)$ since $e^{\pm ikx}$ are linearly independent, therefore we obtain the solutions $C(t) \equiv 0$ and $D(t) = T(k,0)e^{-4ik^3t}$. While these calculations hold in the limit $x \to \pm\infty$ we cannot simply assume that they are still valid when x is not large enough. For x considerably smaller than ∞ the solution is given by the so-called Jost solutions which are denoted by $f_{\pm k}^{\pm}(x,t)$. Jost demonstrated their uniqueness [16]. Hence, we write:

$$\Psi_k(x,t) = A(t)f_{-k}^{+}(x,t) + E(t)f_k^{+}(x,t) = C(t)f_{-k}^{-}(x,t) + D(t)f_k^{-}$$

with $A(t)$, $E(t)$, $C(t)$, $D(t)$ as derived above. It turns out that in the limit x sufficiently large respectively negative the previous calculations do hold. In fact, the time dependent Jost solutions (with the transmission and reflection coefficients) are generally denoted by

$$\psi_k(x,t) = T(k,t)f_k^{-}(x,t) = f_{-k}^{+}(x,t) + R(k,t)f_k^{+}(x,t).$$

Hence, comparing the expressions for $\Psi_k(x,t)$ and $\psi_k(x,t)$ we get $e^{4ik^3t}\Psi_k(x,t) = \psi_k(x,t)$ from which follows that $T(k,t) \equiv T(k,0)$, i.e. the transmission coefficient is time independent, and $R(k,t) = R(k,0)e^{8ik^3t}$ [16]. Finally we have obtained the scattering data at time t

$$S(\lambda,t) = \left(\{\kappa_n, N_n(t)\}_{n=1}^{N}, R(k,t), T(k,t) \right).$$

3. **Inverse Scattering Transform**

We recover the potential $u(x,t)$ by performing the inverse scattering transform. Giving a (mathematical) introduction to this technique is beyond the scope of this report. Hence, we simply postulate the equations required for that. Then we will solve such equations for the special case of a reflectionless potential.

Firstly, using the evolved scattering data we define the function [16]

$$K(\xi,t) := 2\sum_{n=1}^{N} \rho_n(t)e^{-2\kappa_m \xi} + \frac{1}{\pi}\int_{-\infty}^{\infty} R(k,t)e^{2ik\xi}\,\mathrm{d}k. \tag{4.11}$$

Secondly, we solve the (linear) integral equation,

$$K(x+y;t) + B(x,y;t) + \int_0^{\infty} B(x,z;t)K(x+y+z;t)\,\mathrm{d}z = 0 \tag{4.12}$$

which is called the Gel'fand-Levitan-Marchenko equation. From Eq. (4.12) we obtain the function $B(x,y;t)$ and the solution $u(x,t)$ is then obtained by differentiation through the formula:

$$u(x,t) = -\frac{d}{dx}B(x,0;t). \tag{4.13}$$

4.1 Reflectionless Potential

We consider a reflectionless potential, i.e. $R(k,t) \equiv 0$, and in such a case $K(\xi,t)$ reduces to $K(\xi,t) = 2\sum_{n=1}^{N} \rho_n(t)e^{-2\kappa_m\xi}$ with a finite amount of (discrete) eigenvalues κ_n respectively bound states and $\rho_n(t) = \rho_n(0)e^{8\kappa_n^3 t} > 0$. We want to solve the Gel'fand-Levitan-Marchenko equation in order to obtain $B(x,y)$

$$2\sum_{n=1}^{N} \rho_n(t)e^{-2\kappa_m(x+y)} + B(x,y;t) + 2\int_0^\infty B(x,z;t)\sum_{n=1}^{N} \rho_n(t)e^{-2\kappa_m(x+y+z)}\,\mathrm{d}z = 0. \quad (4.14)$$

For that we use the ansatz

$$B(x,y;t) = \sum_{n=1}^{N} \sqrt{\rho_n(t)}e^{-\kappa_n(x+2y)}w_n(x)$$

with $w_n(x)$ to be determined from the Gel'fand-Levitan-Marchenko equation. We will drop writing the explicit time dependence in future steps. By inserting this expression for $B(x,y)$ into Eq. (4.14), we get

$$\sum_{n=1}^{N} e^{-2\kappa_n y}\left\{2\rho_n e^{-2\kappa_n x} + \sqrt{\rho_n}e^{-\kappa_n x}w_n(x) + 2\sum_{m=1}^{N} \rho_n\sqrt{\rho_m}w_m(x)e^{-(2\kappa_n+\kappa_m)x}\int_0^\infty e^{-2(\kappa_n+\kappa_m)z}\,\mathrm{d}z\right\} = 0.$$

Since we have assumed that each eigenvalue κ_i is distinct, the exponentials are linearly independent and never vanish, hence the latter equation is satisfied if the coefficients vanish for every $e^{-2\kappa_n y}$. We further divide by $\sqrt{\rho_n}e^{-\kappa_n x}$ and get after performing the integration

$$2\sqrt{\rho_n}e^{-\kappa_n x} + w_n(x) + 2\sum_{m=1}^{N} \frac{\sqrt{\rho_n\rho_m}}{\kappa_n+\kappa_m}e^{-2(\kappa_n+\kappa_m)x}w_m(x) = 0 \quad \forall n. \quad (4.15)$$

In order to simplify this system of N algebraic equations we define $\boldsymbol{w}(x) = (w_1(x),\cdots,w_N(x))^T$, $\boldsymbol{v}(x) = 2(v_1(x),\cdots,v_N(x))^T$ with $v_i := \sqrt{\rho_i}e^{-\kappa_i x}$, and the symmetric $N\times N$ matrix $S_{ij} = \sqrt{\rho_i\rho_j}e^{-(\kappa_i+\kappa_j)x}/(\kappa_i+\kappa_j)$ for $i,j = 1,2,\cdots N$, besides, let $\mathbb{1}$ denote the identity $N\times N$-matrix. To make sure that the solution is unique, we need to verify that S is positive definite. Consider the quadratic form

$$\langle\boldsymbol{\xi}|S\boldsymbol{\xi}\rangle = \sum_{i,j=1}^{N} \frac{\sqrt{\rho_i}\sqrt{\rho_j}\xi_i\xi_i}{\kappa_i+\kappa_j}e^{-(\kappa_i+\kappa_j)x} = \int_x^\infty \left[\sum_{i=1}^{N} \sqrt{\rho_i}\xi_i e^{-\kappa_i x}\right]^2 \mathrm{d}y \geq 0$$

where $\boldsymbol{\xi} = (\xi_1,\cdots,\xi_N)$. Hence, S is clearly positive definite since the upper term is obviously positive and vanishes only if $\boldsymbol{\xi} = 0$. Using the previous definitions we can rewrite Eq. (4.15) as $\left(\mathbb{1}+S(x)\right)\boldsymbol{w}(x) + \boldsymbol{v}(x) = 0 \Leftrightarrow \boldsymbol{w}(x) = -\left(\mathbb{1}+S(x)\right)^{-1}\boldsymbol{v}(x)$ since $\mathbb{1}+S(x)$ is invertible. Therefore we get

$$B(x,y) = \sum_{n=1}^{N} \sqrt{\rho_n(t)}e^{-\kappa_n}(x+2y)w_n(x) = -\left\langle\left(\mathbb{1}+S(x)\right)^{-1}\boldsymbol{v}(x)\Big|\tfrac{1}{2}\boldsymbol{v}(x+2y)\right\rangle.$$

According to Eq. (4.13) we then set $y = 0$:

$$B(x,0) = -\tfrac{1}{2}\left\langle\left(\mathbb{1}+S(x)\right)^{-1}\boldsymbol{v}(x)\Big|\boldsymbol{v}(x)\right\rangle = -\tfrac{1}{2}\sum_{n,m=1}^{N} \left(\mathbb{1}+S(x)\right)^{-1}_{n,m}v_m(x)v_n(x)$$

$$= 2\sum_{n,m=1}^{N} \left(\mathbb{1}+S(x)\right)^{-1}_{n,m}\frac{d}{dx}\left(\mathbb{1}+S(x)\right)_{m,n} = 2\,\mathrm{tr}\left[\left(\mathbb{1}+S(x)\right)^{-1}\frac{d}{dx}\left(\mathbb{1}+S(x)\right)\right]$$

where we used $\frac{d}{dx}\Big(\mathbb{1} + S(x)\Big)_{n,m} = -\sqrt{\rho_n \rho_m}\, e^{-(\kappa_n + \kappa_m)x} = -\frac{1}{4} v_n(x) v_m(x)$. By using the identity

$$\frac{d}{dx}\Big(\log \det(A(x))\Big) = \mathrm{tr}\Big(A^{-1}(x)\frac{d}{dx}A(x)\Big) \tag{4.16}$$

which we shortly prove in Appendix A, we finally obtain $B(x,0) = 2\frac{d}{dx}\Big(\log \det(\mathbb{1} + S)\Big)$. From the latter expression we deduce the solution by differentiation according to Eq. (4.13):

$$u(x,t) = -2\frac{d^2}{dx^2}\Big(\log \det(\mathbb{1} + S(x))\Big). \tag{4.17}$$

If $R(k,t) \neq 0$, then the Gel'fand-Levitan-Marchenko is not analytically solvable anymore [16]. We would like to emphasize that there exists a one to one correspondence between the number of discrete eigenvalues and the number of solitons which emerge (asymptotically) [14].

Example for N=1

Here we intend to demonstrate how the previously outlined approach can be used to (also) obtain the solitary wave as depicted in Fig. 2. In addition to that, we show the relationship between the number of discrete eigenvalues and the number of solitons. We start by solving the direct scattering problem with the potential $u(x, t = 0) = -2\cosh^{-2}(x)$. The first task is to find the scattering data by solving the (time-independent) Schrödinger equation $\psi_{xx} + \Big(\lambda + 2\cosh^{-2}(x)\Big) = 0$. This latter equation transforms conveniently under the substitution $\zeta := \tanh(x)$ leading to

$$\frac{d}{d\zeta}\Big[\big(1 - \zeta^2\big)\frac{d\psi}{d\zeta}\Big] + \Big[2 + \frac{\lambda}{1 - \zeta^2}\Big]\psi = 0 \tag{4.18}$$

which is immediately recognized as the general Legendre equation with $l = 1$ and $-m^2 := \lambda$, better $m = \kappa_m$. We recall that m may take the values 0 or 1, where we will obtain bounded states only if $m = 1$, respectively $\kappa = \kappa_1 = 1$. The corresponding eigenfunction is the associated Legendre polynomial $\psi_1 \propto P_1^1(\tanh x) = -\cosh^{-1}(x)$. Since $\int_{-\infty}^{\infty}\cosh^{-2}(x)dx = 2$, the normalised eigenfunction becomes $\psi_1(x) = -\frac{1}{\sqrt{2}}\cosh^{-1}(x)$. Analysing the asymptotic behaviour of this solution as $x \to \infty$ we obtain $\psi_1(x) \sim \sqrt{2}e^{-x}$, therefore a scattering data of $N_1(0) = \sqrt{2}$. The time evolved scattering data is then $N_1(t) = \sqrt{2}e^{4t}$, and consequently $\rho_1(t) = 2e^{8t}$ according to the previous results. By inserting the matrix element $S_{11} = e^{8t-2x}$ in Eq. (4.17), we finally get the solution

$$u(x,t) = -2\frac{d^2}{dx^2}\Big(\log\big(1 + e^{8t-2x}\big)\Big) = -2\cosh^{-2}(x - 4t) \tag{4.19}$$

which is precisely the solitary wave obtained in Eq. (4.3) if we set $c = 4$ [1]. More generally, by imposing the initial condition $u(x,0) = -N(N+1)\cosh^{-2}(x)$ and repeating the previous steps we would obtain the N soliton solution. Figure 4 illustrates the (normalised) 2-soliton solution given by

$$u(x,t) = \frac{12}{8} \cdot \frac{3 + 4\cosh(2x - 8t) + \cosh(4x - 64t)}{\Big[3\cosh(x - 28t) + \cosh(3x - 36t)\Big]^2}. \tag{4.20}$$

Such solution could hardly be obtained through a simple ansatz such as the one used at the very beginning of section 4. In Fig. 4 we can see two solitons retain their identities after interacting as described by solution (4.20) [16].
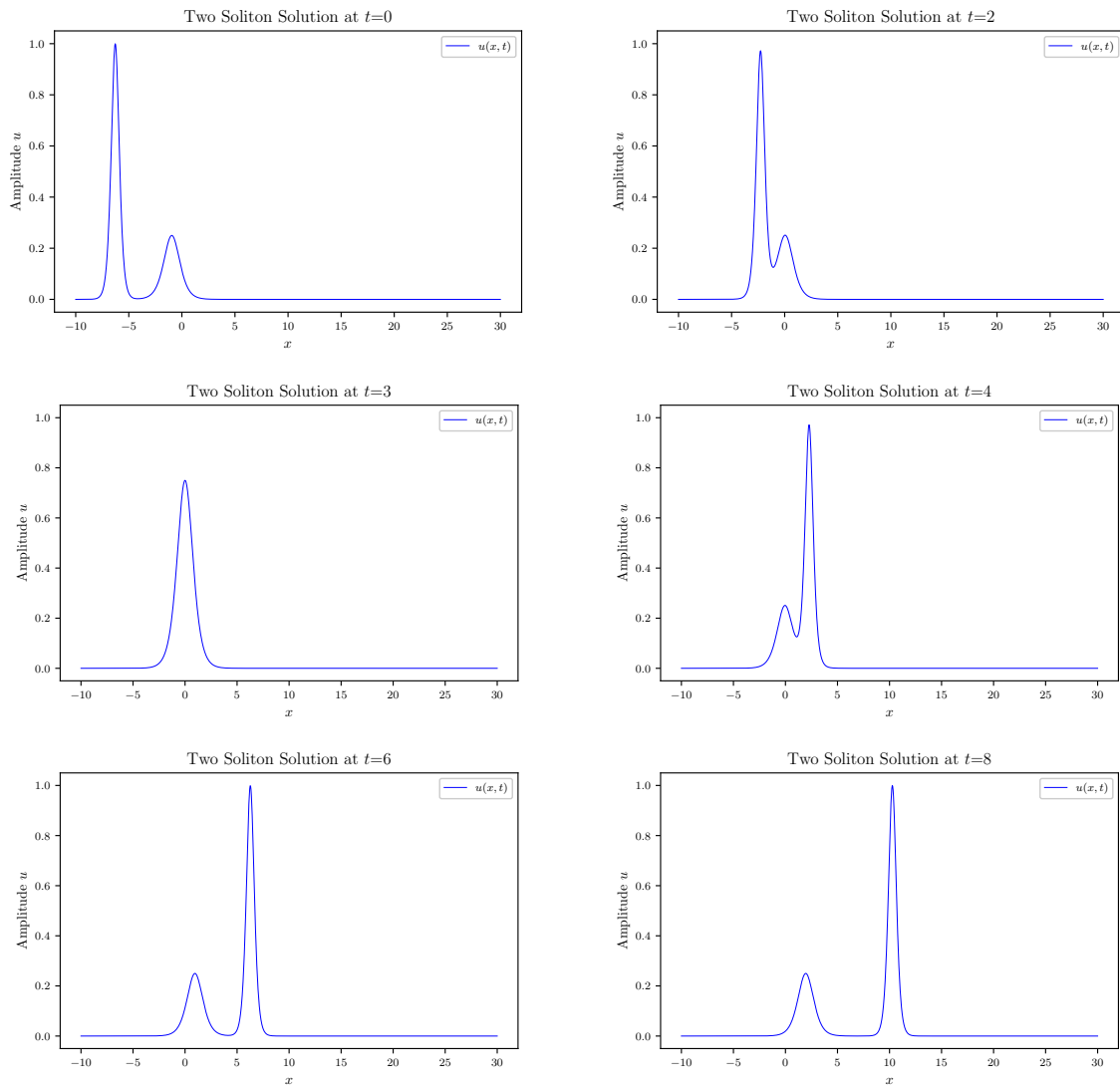
FIGURE 4. Propagation of two solitons according to the solution (4.20).

5 Symmetries of the KdV Equation

A symmetry group of the KdV equation (or any other differential equation) is a group G such that for every $g \in G$ and every solution u of our equation, $g(u)$ is also a solution. The objective of this section is to find such symmetry group [17]. Even though few "ad-hoc" methods exist (which are rather based on guessing an appropriate ansatz) to find some transformations leaving the KdV equation invariant, we will focus on the more general approach using infinitesimal transformations leading to a symmetry group. The great power of these infinitesimal transformations is due to the fact that we can substitute the difficult non-linear conditions for invariance (of a subset or function) under the group transformation with equivalent linear conditions for infinitesimal invariance under the corresponding infinitesimal generators of the group action [18].

In order to find the most general (connected) symmetry group, let

$$\boldsymbol{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{5.1}$$

be a vector field defined on an appropriate space. The numbers p and q denote the amount of dependent and independent variables respectively. We shall interpret such vector field with unknown coefficients $\xi^i(x,u)$ and $\phi_\alpha(x,u)$ as infinitesimal generator of the (Lie-) symmetry group. In other words, the flux generated by this vector field is identified as the symmetry group. In order to determine the coefficients of the vector field $\boldsymbol{v}$ it is necessary to introduce the concept of prolongation of a space. Roughly speaking, the k-th prolongation is the space obtained by adding the k-th mixed derivative. The n-th prolongation of $\boldsymbol{v}$ is then the vector field

$$\mathrm{pr}^{(n)}\boldsymbol{v} = \boldsymbol{v} + \sum_{\alpha=1}^{q} \sum_{J} \phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha} \tag{5.2}$$

with the second summation being over all multi-indices $J = (j_1, \cdots, j_k)$, with $1 \leq j_k \leq p$, $1 \leq k \leq n$. Using the chain rule for derivatives one can show that the coefficients ϕ_α^J of $\mathrm{pr}^{(n)}\boldsymbol{v}$ are given by the following formula:

$$\phi_\alpha^J(x,u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha \tag{5.3}$$

where D_J is the total derivative, $u_i^\alpha := \frac{\partial u^\alpha}{\partial x^i}$ and $u_{J,i}^\alpha := \frac{\partial u_{J,i}^\alpha}{\partial x^i}$. The infinitesimal criterion of invariance is given by the theorem [18]:

Theorem 5.1. *Suppose*

$$\Delta_\nu(x,u^{(n)}) = 0, \quad \nu = 1, \cdots, l$$

is a system of differential equations of maximal rank. If G is a local group of transformation acting on M, and

$$\mathrm{pr}^{(n)}\boldsymbol{v}\Big[\Delta_\nu(x,u^{(n)})\Big] = 0 \quad \nu = 1, \cdots, l, \quad whenever \quad \Delta(x,u^{(n)}) = 0, \tag{5.4}$$

for every infinitesimal generator $\boldsymbol{v}$ of G, then G is a symmetry group of the system.

Proof. See [18], pages 106-107 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will now apply theorem 5.1 in order to find the symmetry group to the KdV equation. For simplicity, we set the constants which appear in the KdV equation to 1 consider the equation[8]

$$u_t + uu_x + u_{xxx} = 0 \tag{5.5}$$

---

[8]Eq. 5.5 is obtained through the rescaling $u \mapsto \frac{1}{6}u$.

which has two independent variables (x, t) and one dependent variable $u(x, t)$. Therefore $p = 2$ and $q = 1$. The KdV is a third order (non-linear) partial differential equation and in that case $n = 3$, hence $\Delta(x, t, u^{(3)}) = u_t + u_{xxx} + uu_x$. We define a vector field according to Eq. (5.1) $\boldsymbol{v} := \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u$. The objective is to determine the coefficients ξ, τ and ϕ so that the vectorfield $\boldsymbol{v}$ generates a (one-parameter) symmetry group of the KdV equation (5.5). According to theorem 5.1 we need to know the third prolongation of $\boldsymbol{v}$. By using Eq. (5.2) and after writing the sums down we get:

$$
\begin{aligned}
\mathrm{pr}^{(3)}\boldsymbol{v} = \boldsymbol{v} &+ \phi^x \frac{\partial}{\partial u_x} + \phi^t \frac{\partial}{\partial u_t} \\
&+ \phi^{xx} \frac{\partial}{\partial u_{xx}} + \phi^{xt} \frac{\partial}{\partial u_{xt}} + \phi^{tt} \frac{\partial}{\partial u_{tt}} \\
&+ \phi^{xxx} \frac{\partial}{\partial u_{xxx}} + \phi^{xxt} \frac{\partial}{\partial u_{xxt}} + \phi^{xtt} \frac{\partial}{\partial u_{xtt}} + \phi^{ttt} \frac{\partial}{\partial u_{ttt}}.
\end{aligned}
$$

The coefficients ϕ^i are computed using Eq. (5.3), where we would like to emphasize that these are cumbersome calculations[9]. We now apply the third prolongation $\mathrm{pr}^{(3)}$ to the KdV equation (5.5), in other words theorem 5.1 infers that the vectorfield $\boldsymbol{v} = \xi\partial_x + \tau\partial_t + \phi\partial_u$ generates a (one-parameter) symmetry group if and only if

$$
\phi^t + \phi^{xxx} + u\phi^x + u_x\phi = 0 \tag{5.6}
$$

where u satisfies Eq. (5.5). By inserting the coefficients ϕ^i, replacing $u_t = -u_{xxx} - uu_x$ (in this way also higher derivatives with respect to time and position are obtained, i.e. u_{xt}, u_{xxt}, etc.) wherever it occurs and isolating each (derivative) monomial, we obtain the defining equations for the symmetry group. Since this is a very long (and tedious) calculation we will not present it here in full detail. As a result, it turns out that the coefficients are given by

$$
\xi = c_1 + c_3 t + c_4 x, \qquad\qquad \tau = c_2 + 3c_4 t, \qquad\qquad \phi = c_3 - 2c_4 u
$$

where c_i are arbitrary (integration) constants. Finally, we collect the linear independent terms of the vectorfield leading to the symmetry (Lie)-algebra $\mathfrak{g}$ of the Korteweg-de Vries equation which is spanned by the four vector (fields):

$$
\begin{aligned}
v_1 &= \partial_x, & v_2 &= \partial_t, \\
v_3 &= t\partial_x + \partial_u, & v_4 &= x\partial_x + 3t\partial_t - 2u\partial_u.
\end{aligned}
$$

It can be shown by elementary calculations that the Lie-bracket closes, i.e. $[v_i, v_j] = c_{ij}^k v_k$ for c_{ij}^k given in Table 1. By exponentiating each generator separately one gets the induced flow, i.e. the symmetry transformation induced by the respective generator: $\tilde{x} = e^{\alpha v_i}x$, $\tilde{t} = e^{\alpha v_i}t$ and $\tilde{u}(\tilde{x}, \tilde{t}) = e^{\alpha v_i}u(x, t)$ for $\alpha \in \mathbb{R}$. See [10] for an explicit calculation. In conclusion, if $u = f(x, t)$

---

[9]As an example we have

$$
\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt} = \phi_x + (\phi_u - \xi_u)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t
$$

and even more complicated

$$
\begin{aligned}
\phi^{xxx} &= D_x^3(\phi - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt} \\
&= D_x^3\phi - u_x D_x^3\xi - u_y D_x^3\tau - 3u_{xx}D_x^2\xi - 3u_{xt}D_x^2\tau - 3u_{xxx}D_x\xi - 3u_{xxt}D_x\tau
\end{aligned}
$$

which becomes a long calculation.
[10]

$$
\begin{aligned}
\tilde{x} = e^{\alpha v_3}x &\quad\rightarrow\quad [1 + \alpha(\partial_u + t\partial_x) + ...]x = x + \alpha t \\
\tilde{t} = e^{\alpha v_3}t &\quad\rightarrow\quad [1 + \alpha(\partial_u + t\partial_x) + ...]t = t \\
\tilde{u} = e^{\alpha v_3}u &\quad\rightarrow\quad [1 + \alpha(\partial_u + t\partial_x) + ...]u = u + \alpha
\end{aligned}
$$

Hence, $\tilde{u}(\tilde{x}, \tilde{t}) = \tilde{u}(x + \alpha t, t) = u(x, t) + \alpha \quad\Leftrightarrow\quad \tilde{u}(x, t) = u(x - \alpha t, t) + \alpha$

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is a solution of the KdV equation, so are

$$u^{(1)} = f(x - \alpha, t) \qquad \text{space translation} \qquad (5.7)$$

$$u^{(2)} = f(x, t - \alpha) \qquad \text{time translation} \qquad (5.8)$$

$$u^{(3)} = f(x - \alpha t, t) + \alpha \qquad \text{Galilean boost} \qquad (5.9)$$

$$u^{(4)} = e^{-2\alpha} f(e^{-\alpha}x, e^{-3\alpha}t) \qquad \text{scaling} \qquad (5.10)$$

Moreover, one can notice that the ansatz $u(x,t) = f(x - ct)$ we made in order to solve the KdV equation, emerged naturally by studying its symmetries. However, the list of symmetries obtained in this way may seem disappointingly small since we previously showed that the KdV equation has an infinite number of conservation laws. The question now is how we would need to modify our approach in order to obtain all possible symmetries. It turns out that it is necessary to generalize the (generating) vector field by allowing its coefficients ξ and ϕ to depend also on derivatives of u. Besides that the methodology remains essentially the same [17], [18].

	v_1	v_2	v_3	v_4
v_1	0	0	0	v_1
v_2	0	0	v_1	$3v_2$
v_3	0	$-v_1$	0	$-2v_3$
v_4	$-v_1$	$-3v_2$	$2v_3$	0

TABLE 1. Represents the Lie bracket $[v_i, v_j]$ where v_i is the i-th column vector and v_j is the j-th row vector [18].

6   Concluding Remarks

In this report we have discussed a non-linear partial differential equation known as the Korteweg-de Vries equation which describes, for instance, the propagation of waves in shallow water. The first remarkable aspect of the KdV equation is that its solutions were found to be in consonance with the experimental observations of solitary waves by J.S. Russell in 1834.

In section 3 we analysed some elegant properties of the KdV equation where we showed that it arises from a 1+1 dimensional field theory. The latter directly leads to the notion of conserved (Noether) current(s). Furthermore, we also showed that a system described by the KdV equation has an infinite amount of conserved quantities.

In section 4 we first demonstrated how Korteweg and de Vries solved their equation before presenting the inverse scattering transform which was introduced by Gardner, Greene, Kruskal and Miura in order to solve special types of non-linear partial differential equations, especially the KdV equation. We verified that a reflectionless potential leads (in the simplest case) to the same solution already obtained by Korteweg and de Vries in 1895 using travelling wave ansatz. We then went a step further by generalising our approach and obtained the two-soliton solution also using the Inverse Scattering Method. Such solution contains some of the main properties characterising solitonic behaviour as, for instance, the stability of the (soliton-)solutions arising from an intrinsic equilibrium between non-linearity and dispersion within the KdV equation and not (directly) from topological reasons.

Lastly, in section 5, we applied Lie theory in order to obtain the symmetry group of the KdV equation. The latter turned out to include, for instance, the geometrical transformations which generated a conserved (Noether) current in section 3. Besides that, we have also shown how the travelling wave ansatz emerge naturally.

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A   Proof of the Identity (4.16)

The identity

$$\frac{d}{dx}\left(\log\det(A(x))\right) = \mathrm{tr}(A^{-1}(x)\frac{d}{dx}A(x))$$

holds for A being an $N \times N$ matrix.

Proof. Let P be a $N \times N$ matrix with (only) positive eigenvalues α_i such that $\log(\det P) = \sum_{i=1}^{N}\log(\alpha_i)$ is well defined. We first show that $\log(\det P) = \mathrm{tr}\log P$. We diagonalize $P = TDT^{-1}$ where D is a diagonal matrix with components α_i; thus, $\mathrm{tr}\log P = \mathrm{tr}\log(TDT^{-1}) = \mathrm{tr}\left(T\log(D)T^{-1}\right) = \sum_{i=1}^{N}\log(\alpha_i)$ since the trace is cyclic. We now return to $\frac{d}{dx}\log(\det P) = \frac{d}{dx}\mathrm{tr}\log P = \mathrm{tr}\left(P^{-1}\frac{dP}{dx}\right)$ since the trace is linear. $\qquad\square$