§ KdV 方程

The KdV equation：

$$\begin{cases} u_t + 6uu_x + u_{xxx} = 0 \\ u(x,0) = f(x) \end{cases}, -\infty < x < \infty, 0 \le t < \infty \quad (1)$$

用以描述淺水波的演進，其中$6uu_x$是非線性的部分，u_{xxx}是相散的部分 (dispersive term)。

Let $\xi = x - ct$ (c represents the wave speed)

Let $u(x,t) = f(x-ct)$，$u_t = \dfrac{df}{d\xi}\dfrac{d\xi}{dt} = -c\dfrac{df}{d\xi}$ then $-cf' + 6ff' + f''' = 0 \ldots (2)$

(2) 積分一次得$-cf + 3f^2 + f'' = A$ 將f'視為積分因子(即兩邊同乘以f')

$$f'f'' = Af' + cff' - 3f^2 f'$$

$$[\frac{1}{2}(\mathrm{f}')^2]' = (Af)' + (\frac{c}{2}f^2)' - (f^3)'$$ 再積分得

$$(f')^2 = 2Af + cf^2 - 2f^3 + B$$

考慮邊界值 $f, f', f'' \to 0$ as $x \to \infty$ then A=B=0

$(f')^2 = cf^2 - 2f^3 = f^2(c - 2f)$，其中$c - 2f > 0$

$\displaystyle\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = \pm\int d\xi$，let $f = \dfrac{c}{2}\mathrm{sech}^2\theta$ then $c - 2f = \ldots = c\tanh^2\theta$

其中

$\cosh x = \dfrac{e^x + e^{-x}}{2}, \sinh x = \dfrac{e^x - e^{-x}}{2}, \cosh^2 x - \sinh^2 x = 1$，$\dfrac{d}{dx}\mathrm{sech}\, x = -\tanh x\, \mathrm{sech}\, x$

$f' = \dfrac{df}{d\xi}, \xi = x - ct$，$df = c\,\mathrm{sech}\,\theta(-\tanh\theta\,\mathrm{sech}\,\theta)d\theta$

$\displaystyle\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = \ldots = \frac{-2\theta}{\sqrt{c}} = \pm(\xi + k) = \pm(\text{x-ct+x}_0)$，$\theta = \dfrac{\sqrt{c}}{2}(x - ct + x_0)$

$u(x,t) = f(x-ct) = \dfrac{c}{2}\mathrm{sech}^2\theta$，where $\theta = \dfrac{\sqrt{c}}{2}(x - ct + x_0)$

§ Proposition (Miura)

If v is a solution to the modified KdV equation $v_t - 6v^2 v_x + v_{xxx} = 0$ then $u = v^2 + v_x$ solves the KdV equation $u_t - 6uu_x + u_{xxx} = 0$。

$$u_t = 2vv_t + v_{xt} = (\frac{\partial}{\partial x} + 2v)v_t$$

$$u_x = 2vv_x + v_{xx}$$

$$u_{xx} = 2v_x^2 + 2vv_{xx} + v_{xxx}$$

$$u_{xxx} = 4v_x v_{xx} + 2v_x v_{xx} + 2vv_{xxx} + v_{xxxx} = 6v_x v_{xx} + (\frac{\partial}{\partial x} + 2v)(v_{xxx})$$

$-6uu_x = -6(v^2 + v_x)(2vv_x + v_{xx})$展開，其中$-6v_x v_{xx}$與上式中的$6v_x v_{xx}$抵銷

$$-6(2v^3 v_x + v^2 v_{xx} + 2vv_x^2) = -6(\frac{\partial}{\partial x} + 2v)(v^2 v_x)$$

We have $(\frac{\partial}{\partial x} + 2v)(v_t - 6v^2 v_x + v_{xxx}) = 0$

若 u 是已知，則$u = v^2 + v_x$是 v 的 Riccati equation。

$v = u_x + u^2$ 稱為 Miura(三浦)transformation

§ Scattering and Inverse Scattering

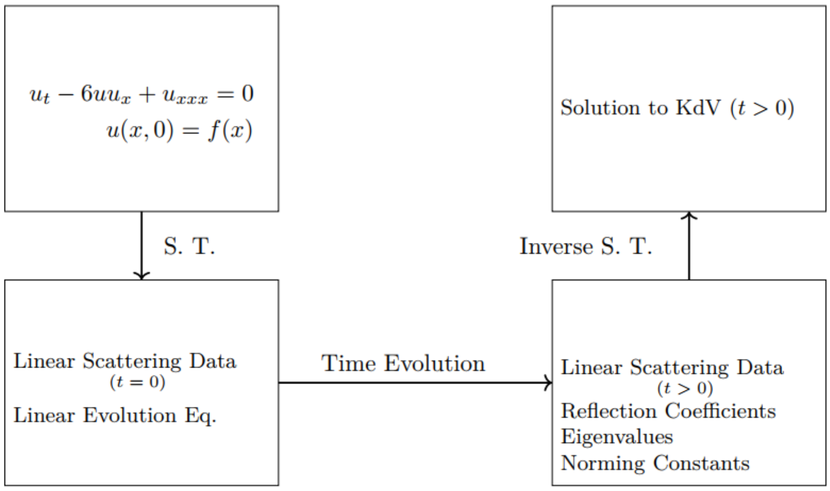

Figure 1: Idea of Scattering and Inverse Scattering

This method transforms the nonlinear problem into a linear problem in the form of a scattering problem，which can be solved more easily。
The solution to the original nonlinear equation is then reconstructed from the scattering data。
A process similar to the Fourier transform。

$u_t + 6uu_x + u_{xxx} = 0$的 Fourier transform 為 $\hat{u}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx$

$u_t = u_{xx}, u(x,0) = f(x), -\infty < x < \infty$
Where u=u(x,t) and f(x) has a Fourier transform。

Define Fourier transform $F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$

and inverse Fourier transform $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)e^{ikx}dk$

$\int_{-\infty}^{\infty}u_t e^{-ikx}dx=\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(ue^{-ikx})dx=\frac{\partial}{\partial t}\int_{-\infty}^{\infty}ue^{-ikx}dx=\frac{\partial U}{\partial t}$ ，where U is the Fourier transform of u(x,t)。

在線性的時候，逆散射變換就是富氏變換。

§ The conservation law　KdV 有各種守恆律

Consider u(x,t)，T=f(u)，X=g(u)　，and u satisfies the equation $\frac{\partial \mathrm{T}}{\partial t}+\frac{\partial \mathrm{X}}{\partial x}=0$ 。

This is a conservation law with density T and flux X。

We have $\frac{d}{dt}\int_{-\infty}^{\infty}Tdx=-X\Big|_{-\infty}^{\infty}=0$　If $X\to 0$ as $|x|\to\infty$

Then $\frac{d}{dt}(\int_{-\infty}^{\infty}Tdx)=0$ ，implies that $\int_{-\infty}^{\infty}Tdx$ =constant。

$0=u_t+6uu_x+u_{xxx}=\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(-3u^2+u_{xx})=0$

$T=u, X=u_{xx}-3u^2$ then $\int_{-\infty}^{\infty}u(x,t)dx$ =constant，where we have taken $u,u_x,u_{xx}\to 0$ as $|x|\to\infty$

$0=u(u_t-6uu_x+u_{xxx})=\frac{\partial}{\partial t}(\frac{1}{2}u^2)+\frac{\partial}{\partial x}(-2u^3+uu_{xx}-\frac{1}{2}u_x^2)$

Take $T=\frac{1}{2}u^2, X=-2u^3+uu_{xx}-\frac{1}{2}u_x^2$

So $\int_{-\infty}^{\infty}u^2=constant$

We define w such that $u=w+\varepsilon w_x+\varepsilon^2 w^2$ ，where u satisfies the KdV equation and ε is any real number。

We can then obtain Gardner equation

$w_t-6(w+\varepsilon^2 w^2)w_x+w_{xxx}=0$ which has a conservation law

$w_t+(-3w^2-3\varepsilon^2 w^3+w_{xx})_x=0$