§ 01 前言 KdV equation is completely integrable。

§ 02 求解 $u(x,t)=f(x-ct)=\frac{c}{2}\text{sech}^2\theta$，where $\theta=\frac{\sqrt{c}}{2}(x-ct+x_0)$

§ 03 Miura transform $u=v^2+v_x$，Riccati equation

§ 04 薛丁格方程 $\psi_{xx}+(\lambda-u)\psi=0$

§ 05 Scattering 散射與反散射 Fourier transform

§ 06 Galilean invariant and Lorentz invariant

§ 07 The conservation law KdV 有各種守恆律

§ 08 KdV soliton

§ 01 前言

故事要從把 1834 年 8 月的某一天說起，蘇格蘭造船工程師 John.Scott Russell 沿著愛丁堡運河策馬前進。發現波浪在行進中保持穩定的速度與形狀，它不會散裂成水面上漂動的浮沫，也不會分流成許多更小的波，不會失去其能量，而只是向前奔流。

KdV 方程指的是是荷蘭數學家 Jojannes Korteweg 與 Gustav de Vries 於 1895 年在推導淺水波方程(shallow water wave)的過程中共同發現的一種偏微分方程。

以下摘自[孤立波淺談]，並參考 Rudy Lee Horne 的[孤粒子簡介]，只是把細節補充一下。

§ 02 推導

The KdV equation：

$$\begin{cases} u_t+6uu_x+u_{xxx}=0 \\ u(x,0)=f(x) \end{cases},-\infty<x<\infty,0\le t<\infty \quad (1)$$

用以描述淺水波的演進，其中$6uu_x$是非線性的部分，u_{xxx}是相散的部分 (dispersive term)。

Let $\xi=x-ct$ (c represents the wave speed)

Let $u(x,t)=f(x-ct)$，$u_t=\frac{df}{d\xi}\frac{d\xi}{dt}=-c\frac{df}{d\xi}$ then $-cf'+6ff'+f'''=0...(2)$

(2) 積分一次得$-cf+3f^2+f''=A$ 將 f'視為積分因子(即兩邊同乘以 f')

$f'f''=Af'+cff'-3f^2f'$

$[\frac{1}{2}(f')^2]'=(Af)'+(\frac{c}{2}f^2)'-(f^3)'$ 再積分得

$(f')^2=2Af+cf^2-2f^3+B$

考慮邊界值 $f,f',f''\to 0$ as $x\to\infty$ then A=B=0

$(f')^2=cf^2-2f^3=f^2(c-2f)$，其中$c-2f>0$

$$\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = \pm\int d\xi$$ ，let $f = \frac{c}{2}\operatorname{sech}^2\theta$ then $c-2f = \ldots = c\tanh^2\theta$

其中

$\cosh x = \frac{e^x+e^{-x}}{2}, \sinh x = \frac{e^x-e^{-x}}{2}, \cosh^2 x - \sinh^2 x = 1$ ，$\frac{d}{dx}\operatorname{sech} x = -\tanh x \operatorname{sech} x$

$f' = \frac{df}{d\xi}, \xi = x - ct$ ，$df = c\operatorname{sech}\theta(-\tanh\theta\operatorname{sech}\theta)d\theta$

$\int \frac{df}{f(c-2f)^{\frac{1}{2}}} = \ldots = \frac{-2\theta}{\sqrt{c}} = \pm(\xi+k) = \pm(\text{x-ct+x}_0)$ ，$\theta = \frac{\sqrt{c}}{2}(x-ct+x_0)$

$u(x,t) = f(x-ct) = \frac{c}{2}\operatorname{sech}^2\theta$ ，where $\theta = \frac{\sqrt{c}}{2}(x-ct+x_0)$

§ 03 Proposition (Miura)

If v is a solution to the modified KdV equation $v_t - 6v^2v_x + v_{xxx} = 0$ then $u = v^2 + v_x$ solves the KdV equation $u_t - 6uu_x + u_{xxx} = 0$ 。

$u_t = 2vv_t + v_{xt} = (\frac{\partial}{\partial x} + 2v)v_t$

$u_x = 2vv_x + v_{xx}$

$u_{xx} = 2v_x^2 + 2vv_{xx} + v_{xxx}$

$u_{xxx} = 4v_xv_{xx} + 2v_xv_{xx} + 2vv_{xxx} + v_{xxxx} = 6v_xv_{xx} + (\frac{\partial}{\partial x} + 2v)(v_{xxx})$

$-6uu_x = -6(v^2+v_x)(2vv_x+v_{xx})$ 展開，其中 $-6v_xv_{xx}$ 與上式中的 $6v_xv_{xx}$ 抵銷

$-6(2v^3v_x + v^2v_{xx} + 2vv_x^2) = -6(\frac{\partial}{\partial x} + 2v)(v^2v_x)$

We have $(\frac{\partial}{\partial x} + 2v)(v_t - 6v^2v_x + v_{xxx}) = 0$

若 u 是已知，則 $u = v^2 + v_x$ 是 v 的 Riccati equation 。

$v = u_x + u^2$ 稱為 Miura(三浦)transformation

§ 04 薛丁格方程

Let $v = \frac{\psi'}{\psi}$ ，then (3)變成 $\psi_{xx} - u\psi = 0$ here $\psi' = \frac{d\psi}{dx}$

可把此式改寫成

$\psi_{xx} + (\lambda - u)\psi = 0$ $\quad Schr\ddot{o}dinger$ equation

Where u(x,t) plays the role of a potential and λ is an eigenvalue of $\psi(x,t)$

§ 05 Scattering and Inverse Scattering

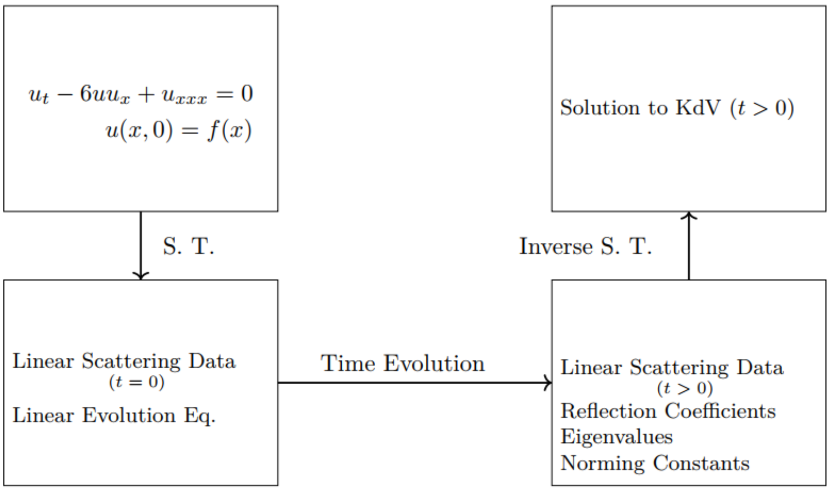

Figure 1: Idea of Scattering and Inverse Scattering

A process similar to the Fourier transform。

Inverse scattering transform(逆散射變換)理論 for KdV

1967 年 Gardner Greene Martin Kruskal Robert Miura

$u_t + 6uu_x + u_{xxx} = 0$ 的 Fourier transform 為

$$\hat{u}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx$$

1974 年把此方法推廣 稱 AKNS scheme 首先解了 sine-Gordon 方程。

If one is given a linear pde with some initial condition, then the solution of the linear pde can be determine using the following steps:

- take the Fourier transform of the linear pde which results in a linear ordinary differential equation (ode) in the Fourier space
- take the Fourier transform of the initial condition (usually not too difficult)
- solve the resulting ode with its initial condition in Fourier space
- transform back to physical space and obtain the solution in the original variables.

$u_t = u_{xx}, u(x,0) = f(x), -\infty < x < \infty$

Where u=u(x,t) and f(x) has a Fourier transform。

Define Fourier transform $F(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx$

and inverse Fourier transform $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)e^{ikx}dk$

$\int_{-\infty}^{\infty}u_t e^{-ikx}dx=\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(ue^{-ikx})dx=\frac{\partial}{\partial t}\int_{-\infty}^{\infty}ue^{-ikx}dx=\frac{\partial U}{\partial t}$，where U is the Fourier transform of u(x,t)

在線性的時候，逆散射變換就是富氏變換。

§ 06 Galilean invariant

Galilean transformation：

A uniform motion $(x,t)\rightarrow(x+tv,t)$

A translation $(x,t)\rightarrow(x+a,t+s)$

A rotation $(x,t)\rightarrow(Rx,t)$

The transformation which describes Galilean invariance is given by：

x=$x'+6\lambda t',t=t',u=u'-\lambda,-\infty<\lambda<\infty$ ..(*)

…

$\tilde{u}_T+6\tilde{u}\tilde{u}_X+\tilde{u}_{XXX}=0$

The KdV equation is invariant under the transformation given by (*)。

But it is not Lorentz-invariant。

在場論中與孤立子連結的問題中，最複雜的當屬相對不變量(relativistic invariants)的發現。所謂相對不變是指經過羅倫茲變換之後保持原形式。

Lorentz transfomation

$t'=\gamma(t-\frac{vx}{c^2}),x'=\gamma(x-vt),y'=y,z'=z$

Where c is the speed of light，and $\gamma=\frac{1}{\sqrt{1-v^2/c^2}}$

§ 07 The conservation law　KdV 有各種守恆律

Consider u(x,t)，T=f(u)，X=g(u) ，and u satisfies the equation $\frac{\partial T}{\partial t}+\frac{\partial X}{\partial x}=0$。

This is a conservation law with density T and flux X。

We have $\frac{d}{dt}\int_{-\infty}^{\infty}Tdx=-X\Big|_{-\infty}^{\infty}=0$ If $X\rightarrow 0$ as $|x|\rightarrow\infty$

Then $\frac{d}{dt}(\int_{-\infty}^{\infty}Tdx)=0$，implies that $\int_{-\infty}^{\infty}Tdx$=constant。

$$0 = u_t + 6uu_x + u_{xxx} = \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(-3u^2 + u_{xx}) = 0$$

$T = u, X = u_{xx} - 3u^2$ then $\int_{-\infty}^{\infty} u(x,t)dx$ =constant，where we have taken $u, u_x, u_{xx} \to 0$ as $|x| \to \infty$

$$0 = u(u_t - 6uu_x + u_{xxx}) = \frac{\partial}{\partial t}(\frac{1}{2}u^2) + \frac{\partial}{\partial x}(-2u^3 + uu_{xx} - \frac{1}{2}u_x^2)$$

Take $T = \frac{1}{2}u^2, X = -2u^3 + uu_{xx} - \frac{1}{2}u_x^2$

So $\int_{-\infty}^{\infty} u^2 = constant$

We define w such that $u = w + \varepsilon w_x + \varepsilon^2 w^2$，where u satisfies the KdV equation and ε is any real number。

We can then obtain Gardner equation

$w_t - 6(w + \varepsilon^2 w^2)w_x + w_{xxx} = 0$ which has a conservation law

$$w_t + (-3w^2 - 3\varepsilon^2 w^3 + w_{xx})_x = 0$$

§ 08 KdV soliton 動畫

The soliton solution of KdV equation is $u(x) = -u_0 \operatorname{sech}^2(x)$

$$\frac{\partial \eta}{\partial \tau} + \frac{3}{2}\sqrt{\frac{g}{h}}\frac{\partial}{\partial \xi}(\frac{1}{2}\eta^2 + \frac{2}{3}\alpha\eta + \frac{1}{3}\sigma\frac{\partial^2 \eta}{\partial \xi^2}) = 0$$

η is the surface elevation (of the wave) above the equilibrium level h，α a constant related to yhe uniform motion of the liquid，g is the gravitational acceleration and $\sigma := \frac{1}{3}h^3 - \frac{Th}{\rho g}$ with T denoting the surface tension and ρ the (liquid) density。

…

$$u_t + 6uu_x + u_{xxx} = 0$$

[Chaotic dynamics in wave motion]

In scaled variables and a fixed refernce frame KdV takes the following form

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = 0 \quad \text{where} \quad \alpha = \frac{A}{H}, \beta = (\frac{H}{L})^2$$

Where $\eta(x,t)$ represents the wave profile，A is wave amplitude，L its average wavelength and H is water depth。

KdV2：extended KdV equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} - \frac{3}{8}\alpha^2\eta^2\eta_x + \alpha\beta\left(\frac{23}{24}\eta_x\eta_{xx} + \frac{5}{12}\eta\eta_{xxx}\right) + \frac{19}{360}\beta^2\eta_{xxxxx} = 0. \quad (2)$$

Soliton solution of KdV2

$\eta(x,t) = A\text{sech}[B(x - vt)]$