§

Example. In a space with coordinates x, y, and z we consider the field of planes given by the equation $\mathrm{d}z = y\,\mathrm{d}x$. (This gives a linear equation for the coordinates of the tangent vector at each point, and that equation determines a plane.)

Problem 1. Draw this field of planes and prove that it has no integral surface, that is, no surface whose tangent plane at every point coincides with the plane of the field.

[Frobenius 可積定理]

$\omega = ydx - dz$ ， $d\omega = dy \wedge dx$ ， $d\omega \wedge \omega = dx \wedge dy \wedge dz \neq 0$

所以 $\omega = 0$ 是不可積的。

對於一個 1-form ω，存在函數 f、g 使得 $\omega = fdg$ 的條件是什麼？

換句話說，要找 $\omega = 0$ 的積分因子。此時積分曲面即為 g=constant。

例 $\omega = yzdx + xzdy + dz$

$d\omega = zdy \wedge dx + ydz \wedge dx + zdx \wedge dy + xdz \wedge dy = ydz \wedge dx + xdz \wedge dy$

$d\omega \wedge \omega = 0$

假設 $\omega = fdg = f(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dz)$ 則

$$\begin{cases} f\frac{\partial g}{\partial x} = yz \\ f\frac{\partial g}{\partial y} = xz \\ f\frac{\partial g}{\partial z} = 1 \end{cases}$$

由(1)(2) $x\frac{\partial g}{\partial x} - y\frac{\partial g}{\partial y} = 0$ ，has a general solution $g = h(z)e^{xy}$

Then $f\frac{\partial g}{\partial z} = fe^{xy}h'(z) = 1$ ， $f = e^{-xy}, h(z) = z$ 所以積分曲面是 $ze^{xy} = constant$

(即 g=constant)