Peter J. Olver

Ch9 A General Framework for Linear PDE

9.1 Adjoints

9.1.1. Choose one from the following list of inner products on $\mathbb{R}^2$. Then find the adjoint of $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ when your inner product is used on both its domain and target space. (a) The Euclidean dot product; (b) the weighted inner product $\langle \mathbf{v}, \mathbf{w} \rangle = 2 v_1 w_1 + 3 v_2 w_2$; (c) the inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w}$ defined by the symmetric positive definite matrix $C = \begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$.

9.1.2. From the list in Exercise 9.1.1, choose a different inner product on the domain and the target space, and then determine the adjoint of the matrix A.

9.1.3. Choose one from the following list of inner products on $\mathbb{R}^3$ for both the domain and target space, and find the adjoint of $A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix}$. (a) The Euclidean dot product on $\mathbb{R}^3$; (b) the weighted inner product $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + 2 v_2 w_2 + 3 v_3 w_3$; (c) the inner product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T C \mathbf{w}$ defined by the symmetric positive definite matrix $C = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$.

9.1.4. From the list in Exercise 9.1.3, choose different inner products on the domain and target space, and then compute the adjoint of the matrix A.

9.1.5. Choose an inner product on $\mathbb{R}^2$ from the list in Exercise 9.1.1 and an inner product on $\mathbb{R}^3$ from the list in Exercise 9.1.3, and then compute the adjoint of $A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 1 \end{pmatrix}$.

9.1.6. (a) Let C be an $m \times n$ matrix. Suppose $\mathbf{u}^T C \mathbf{v} = 0$ for all $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$. Prove that $C = \mathrm{O}$ must be the zero matrix. (b) Let A, B be $m \times n$ matrices such that $\mathbf{u}^T A \mathbf{v} = \mathbf{u}^T B \mathbf{v}$ for all $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$. Prove that $A = B$. (c) Find an $n \times n$ matrix $C \neq \mathrm{O}$ such that $\mathbf{u}^T C \mathbf{u} = 0$ for all $\mathbf{u} \in \mathbb{R}^n$.

9.1.7. Let $U = \mathrm{C}^0[0,1]$. Find the adjoint I^* of the identity operator $\mathrm{I}: U \to U$ under the weighted inner products (9.16).

9.1.8. Compute the adjoint of the derivative operator $v = D[u] = u'$ under the weighted inner products $\langle u, \tilde{u} \rangle = \int_0^1 e^x\, u(x)\, \tilde{u}(x)\, dx$, $\langle\langle v, \tilde{v} \rangle\rangle = \int_0^1 (1+x)\, v(x)\, \tilde{v}(x)\, dx$. Clearly state any boundary conditions that you are imposing.

9.1.9. Let $L[u] = x\,u'(x) + u(x)$ and $0 < a < x < b$. When subject to homogeneous Dirichlet boundary conditions $u(a) = u(b) = 0$, determine the adjoint $L^*[v]$ with respect to
(a) the L^2 inner products (9.7); (b) the weighted inner products (9.16).

9.1.10. Consider the linear operator $L[u] = \begin{pmatrix} u' \\ u \end{pmatrix}$ that maps $u(x) \in C^1$ to the vector-valued function whose components consist of the function and its first derivative. Imposing the boundary conditions $u(0) = u(1)$, compute the adjoint L^* with respect to the L^2 inner products on both the domain and target spaces.

9.1.11. *True or false*: The adjoint of the divergence operator $\nabla \cdot \mathbf{v}$ with respect to the L^2 inner products (9.22, 23) is minus the gradient operator: $(\nabla\cdot)^* u = -\nabla u$. If true, what boundary conditions do you need to assume? If false, what is the adjoint?

9.1.12. Find the adjoint of the two-dimensional curl operator $\nabla \times \mathbf{v}$, as defined in (6.73), with respect to the L^2 inner products (9.22, 23). Carefully state any required boundary conditions.

9.1.13. Prove that (a) the adjoint of a linear operator is also a linear operator;
(b) the adjoint is unique.

9.1.14. Let $L, M: U \to V$ be linear operators on the same inner product spaces. Prove that
(a) $(L + M)^* = L^* + M^*$, (b) $(c\,L)^* = c\,L^*$ for $c \in \mathbb{R}$.

9.1.15. Prove Proposition 9.5.

9.1.16. Prove Proposition 9.6.

9.1.17. *True or false*: If $L: U \to U$ is invertible, then $(L^{-1})^* = (L^*)^{-1}$.

9.1.18. Use the Fredholm Alternative to determine whether the following linear systems are compatible. When compatible, write down the general solution.

(a) $\begin{aligned} 2x - 4y &= -2, \\ -x + 2y &= 3, \end{aligned}$ (b) $\begin{aligned} 6x - 3y + 9z &= 6, \\ 2x - y + 3z &= 2, \end{aligned}$ (c) $\begin{aligned} 2x + 3y &= -1, \\ 3x + 7y &= 1, \\ x + 4y &= 2, \\ -x + y &= 3, \end{aligned}$

(d) $\begin{aligned} 2x_1 - 3x_2 - x_3 &= -1, \\ 3x_1 - x_2 &= 1, \\ 4x_1 + x_2 + x_3 &= 2, \end{aligned}$ (e) $\begin{aligned} 2x_1 + 3x_2 - x_4 &= -1, \\ 3x_1 + 2x_3 - x_4 &= 0, \\ x_1 - x_2 + x_3 &= 1. \end{aligned}$

9.1.19. Use the Fredholm Alternative to find the compatibility conditions for the following systems of linear equations.
(a) $2x + y = a,\ \ x + 4y = b,\ \ -3x + 2y = c$;
(b) $x + 2y + 3z = a,\ \ -x + 5y - 2z = b,\ \ 2x - 3y + 5z = c$;
(c) $x_1 + 2x_2 + 3x_3 = b_1,\ \ x_2 + 2x_3 = b_2,\ \ 3x_1 + 5x_2 + 7x_3 = b_3,\ \ -2x_1 + x_2 + 4x_3 = b_4$;
(d) $x - 3y + 2z + w = a,\ \ 4x - 2y + 2z + 3w = b,\ \ 5x - 5y + 4z + 4w = c,\ \ 2x + 4y - 2z + w = d$.

9.1.20. Suppose A is a symmetric matrix. Show that the linear system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is orthogonal to $\ker A$.

9.1.21. Use the Fredholm Alternative to determine whether there exists a solution to the following boundary value problem: $x u'' + u' = 1 - \frac{2}{3}x,\ \ u'(0) = u'(1) = 0$. If so, write down all solutions.

9.1.22. Analyze the periodic boundary value problem $-u'' = f(x),\ u(0) = u(2\pi),\ u'(0) = u'(2\pi)$, along the same lines as in Example 9.12. Characterize the forcing functions for which the problem has a solution. Explain why the constraints, if any, are in accordance with the Fredholm Alternative. Write down a forcing function $f(x)$ that satisfies all your constraints, and then find all corresponding solutions.

9.1.23. Answer Exercise 9.1.22 for the boundary value problems:
(a) $u'''' = f(x),\quad u''(0) = u'''(0) = 0,\quad u''(1) = u'''(1) = 0$;
(b) $u'''' = f(x),\quad u''(0) = u'''(0) = 0,\quad u(1) = u''(1) = 0$.

9.1.24. Let λ be a real parameter. (a) For which values of λ does the boundary value problem $u'' + \lambda u = h(x),\ u(0) = 0,\ u(1) = 0$, have a unique solution? (b) Construct the Green's function for all such λ. (c) In the nonunique cases, use the Fredholm Alternative to find conditions on the forcing function $h(x)$ that are required for the existence of a solution.

9.1.25. Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected domain. Using the L^2 inner products (9.22, 23) on scalar and vector fields, write out the Fredholm Alternative constraints for the solvability of the boundary value problem $\nabla \cdot \mathbf{v} = f$ in Ω, subject to the homogeneous boundary conditions $\mathbf{v} \cdot \mathbf{n} = 0$ on $\partial\Omega$.

9.1.26. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain. Using the L^2 inner products (9.22, 23) on scalar and vector fields on a domain $\Omega \subset \mathbb{R}^2$, write out the Fredholm Alternative constraints for the solvability of the boundary value problem $\nabla u = \mathbf{f}$ in Ω, subject to the homogeneous boundary conditions $u = 0$ on $\partial\Omega$.

9.2 Self-Adjoint and Positive Definite Linear Functions

9.2.1. Which of the following matrices define self-adjoint linear functions $S: \mathbb{R}^2 \to \mathbb{R}^2$ relative to the dot product? (a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, (b) $\begin{pmatrix} 0 & 3 \\ 2 & 2 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & 0 \\ 2 & -5 \end{pmatrix}$, (d) $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$.

9.2.2. Answer Exercise 9.2.1 for the inner products

$$(i)\ \langle \mathbf{u}, \widetilde{\mathbf{u}} \rangle = 2u_1\widetilde{u}_1 + 3u_2\widetilde{u}_2; \qquad (ii)\ \langle \mathbf{u}, \widetilde{\mathbf{u}} \rangle = \mathbf{u}^T C \widetilde{\mathbf{u}}, \text{ where } C = \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}.$$

9.2.3. *True or false*: Given an inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ on $\mathbb{R}^n$:
(*a*) The inverse of a nonsingular self-adjoint $n \times n$ matrix is self-adjoint.
(*b*) The inverse of a nonsingular positive definite $n \times n$ matrix is positive definite.

9.2.4. Prove that $K > 0$ is a positive definite $n \times n$ matrix if and only if $J = K^T + K$ is a symmetric positive definite matrix.

9.2.5. (a) Prove that the $n \times n$ matrix K defines a self-adjoint linear function on $\mathbb{R}^n$ with respect to the inner product $\langle \mathbf{u}, \widetilde{\mathbf{u}} \rangle = \mathbf{u}^T C \widetilde{\mathbf{u}}$ for C a symmetric positive definite matrix if and only if the matrix $J = CK$ is symmetric, and hence defines a self-adjoint linear function with respect to the dot product. (*b*) Prove that $K > 0$ under the given inner product if and only if $J > 0$ under the dot product.

9.2.6. Let $D[u] = u'$ be the derivative operator acting on the vector space of C^2 scalar functions $u(x)$ defined for $0 \le x \le 1$ and satisfying the boundary conditions $u(0) = 0$, $u(1) = 0$.
(a) Given the weighted inner product $\langle u, \widetilde{u} \rangle = \int_0^1 u(x)\,\widetilde{u}(x)\,e^x\,dx$ on both its domain and target spaces, determine the corresponding adjoint operator D^*.
(*b*) Let $S = D^* \circ D$. Write down and solve the boundary value problem $S[u] = 2e^x$.

9.2.7. Let $c(x) \in \mathrm{C}^0[a, b]$ be a continuous function. Prove that the linear multiplication operator $S[u] = c(x)\,u(x)$ is self-adjoint with respect to the L^2 inner product. What sort of boundary conditions need to be imposed?

9.2.8. *True or false*: The Neumann boundary value problem $-u'' + u = x$, $u'(0) = u'(\pi) = 0$, admits a unique solution.

9.2.9. Prove that the complex differential operator $L[u] = \mathrm{i}\,\dfrac{du}{dx}$ is self-adjoint with respect to the L^2 Hermitian inner product $\langle u, v \rangle = \int_{-\pi}^{\pi} u(x)\,\overline{v(x)}\,dx$ on the space of continuously differentiable complex-valued 2π–periodic functions: $u(x + 2\pi) = u(x)$.

9.2.10. Let $L = D^2$. Using the L^2 inner products on both the domain and target spaces, write down a set of homogeneous boundary conditions that makes $L^* = D^2$. Then set $S = L^* \circ L = D^4$. Do your boundary conditions lead to a boundary value problem $S[u] = f$ that is (i) positive definite; (ii) positive semi-definite; or (iii) neither?

9.2.11. Let β be a real constant. *True or false*: The second derivative operator $S[u] = u''$ is self-adjoint with respect to the L^2 inner product on the space of functions

$$U = \left\{ u(x) \in C^2[0,1] \;\middle|\; u(0) = 0,\ u'(1) + \beta\, u(1) = 0 \right\}$$

subject to Dirichlet boundary conditions at the left-hand endpoint and Robin boundary conditions at the right-hand endpoint.

9.2.12. Let β be a real constant. Consider the differential operator $S[u] = -u''$ acting on the space of functions

$$U = \left\{ u(x) \in C^2[0,1] \;\middle|\; u(0) = 0,\ u'(1) + \beta\, u(1) = 0 \right\}$$

subject to Dirichlet boundary conditions at the left-hand endpoint and Robin boundary conditions at the right-hand endpoint. Prove that $S > 0$ is positive definite with respect to the L^2 inner product if and only if $\beta > -1$. *Hint*: Use the analysis following (4.48).

9.2.13. The equilibrium equations for a toroidal membrane (an inner tube) lead to the Poisson equation $-u_{xx} - u_{yy} = f(x,y)$ on a rectangle $0 < x < a$, $0 < y < b$, subject to periodic boundary conditions

$$u(x,0) = u(x,b), \qquad u_y(x,0) = u_y(x,b), \qquad u(0,y) = u(a,y), \qquad u_x(0,y) = u_x(a,y).$$

(a) Prove that the toroidal boundary value problem is self-adjoint. (b) Is it positive definite, positive semi-definite, or neither? (c) Are there any conditions that must be imposed on the forcing function $f(x,y)$ in order that a solution exist?

9.2.14. Find the adjoint of the gradient operator ∇ with respect to the L^2 inner product (9.22) between scalar fields, and the following weighted inner product between (column) vector fields $\mathbf{v} = (\, v_1(x,y), v_2(x,y)\,)^T$, $\tilde{\mathbf{v}} = (\, \tilde{v}_1(x,y), \tilde{v}_2(x,y)\,)^T$:

$$\langle\!\langle \mathbf{v}\,, \tilde{\mathbf{v}} \rangle\!\rangle = \iint_\Omega \mathbf{v}(x,y)^T C(x,y)\, \tilde{\mathbf{v}}(x,y)\, dx\, dy,$$

where the 2×2 matrix $C(x,y) = \begin{pmatrix} \alpha(x,y) & \beta(x,y) \\ \beta(x,y) & \gamma(x,y) \end{pmatrix} > 0$ is symmetric, positive definite at all points $(x,y) \in \Omega$. What sort of boundary conditions do you need to impose? Write out the corresponding boundary value problem for the equilibrium equation $\nabla^* \circ \nabla u = f$.

9.2.15. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Construct a set of homogeneous boundary conditions on $\partial\Omega$ that make the *biharmonic equation* $\Delta^2 u = f$: (a) self-adjoint, (b) positive definite, (c) positive semi-definite, but not positive definite.

9.2.16. Write down the boundary value problem $\widehat{S}_\xi[\widehat{G}_\xi] = \delta_\xi$ satisfied by the modified Green's function $\widehat{G}_\xi(x) = \widehat{G}(x;\xi)$ given in (9.59). Is the underlying linear operator $\widehat{S}_\xi$, which may depend on ξ, self-adjoint with respect to a suitable inner product?

9.2.17. Prove symmetry of the Green's function, $G(\boldsymbol{\xi};\mathbf{x}) = G(\mathbf{x};\boldsymbol{\xi})$, for the Poisson equation on a bounded domain $\Omega \subset \mathbb{R}^2$ subject to homogeneous Dirichlet boundary conditions. *Hint*: Look at how we established (9.56).

9.2.18. Generalize Exercise 9.2.17 to the partial differential equation (9.61).

9.3 Minimization Prinliples

9.3.1. Consider the boundary value problem $-u'' = x$, $u(0) = u(1) = 0$. (*a*) Find the solution. (*b*) Write down a minimization principle that characterizes the solution. (*c*) What is the value of the minimized quadratic functional on the solution? (*d*) Write down at least two other functions that satisfy the boundary conditions and check that they produce larger values for the energy.

9.3.2. Answer Exercise 9.3.1 for the boundary value problems
(*a*) $\dfrac{d}{dx}\left(\dfrac{1}{1+x^2}\,\dfrac{du}{dx}\right) = x^2$, $u(-1) = u(1) = 0$; (*b*) $-(e^x u')' = e^{-x}$, $u(0) = u'(1) = 0$;
(*c*) $x^2 u'' + 2xu' = 3x^2$, $u'(1) = u(2) = 0$; (*d*) $xu'' + 3u' = 1$, $u(-2) = u(-1) = 0$.

9.3.3. Let $Q[u] = \displaystyle\int_0^1 \left[\tfrac{1}{2}(u')^2 - 5u\right] dx$. (*a*) Find the function $u_\star(x)$ that minimizes $Q[u]$ among all C^2 functions that satisfy $u(0) = u(1) = 0$.
(*b*) Test your answer by computing $Q[u_\star]$ and then comparing with the value of $Q[u]$ when $u(x) =$ (*i*) $x - x^2$, (*ii*) $\frac{3}{2}x - \frac{3}{2}x^3$, (*iii*) $\frac{2}{3}\sin \pi x$, (*iv*) $x^2 - x^4$.

9.3.4. For each of the following functionals and associated boundary conditions: (*i*) write down a boundary value problem satisfied by the minimizing function, and (*ii*) find the minimizing function $u_\star(x)$:

(a) $\displaystyle\int_0^1 \left[\tfrac{1}{2}(u')^2 - 3u\right] dx$, $u(0) = u(1) = 0$,

(b) $\displaystyle\int_0^1 \left[\tfrac{1}{2}(x+1)(u')^2 - 5u\right] dx$, $u(0) = u(1) = 0$,

(c) $\displaystyle\int_1^3 \left[x(u')^2 + 2u\right] dx$, $u(1) = u(3) = 0$,

(d) $\displaystyle\int_0^1 \left[\tfrac{1}{2}e^x(u')^2 - (1+e^x)u\right] dx$, $u(0) = u(1) = 0$,

(e) $\displaystyle\int_{-1}^1 \frac{(x^2+1)(u')^2 + xu}{(x^2+1)^2}\, dx$, $u(-1) = u(1) = 0$.

9.3.5. Which of the following quadratic functionals possess a unique minimizer among all C^2 functions satisfying the indicated boundary conditions? Find the minimizer if it exists.

(a) $\int_1^2 \left[\frac{1}{2}x\,(u')^2 + 2\,(x-1)\,u\right] dx, \quad u(1) = u(2) = 0;$

(b) $\int_{-\pi}^{\pi} \left[\frac{1}{2}x\,(u')^2 - u\cos x\right] dx, \quad u(-\pi) = u(\pi) = 0;$

(c) $\int_{-1}^{1} \left[(u')^2\cos x - u\sin x\right] dx, \quad u(-1) = u'(1) = 0;$

(d) $\int_{-2}^{2} \left[(1-x^2)\,(u')^2 - u\right] dx, \quad u(-2) = u(2) = 0;$

(e) $\int_0^1 \left[(x+1)\,(u')^2 - u\right] dx, \quad u'(0) = u'(1) = 0.$

9.3.6. Let $D[u] = u'$ be the derivative operator acting on the vector space of C^2 scalar functions $u(x)$ defined for $0 \le x \le 1$ and satisfying the boundary conditions $u(0) = 0$, $u'(1) = 0$.

(a) Given the weighted inner product $\langle u, \tilde{u} \rangle = \int_0^1 u(x)\,\tilde{u}(x)\,e^x\,dx$ on both its domain and target spaces, determine the corresponding adjoint operator D^*.

(b) Let $S = D^* \circ D$. Write down and solve the boundary value problem $S[u] = 3\,e^x$.

(c) Write down a minimization principle that characterizes the solution you found in part (b), or explain why none exists.

9.3.7. Solve the Sturm–Liouville boundary value problem $-4u'' + 9u = 1$, $u(0) = 0$, $u(2) = 0$. Is your solution unique?

9.3.8. Answer Exercise 9.3.7 for the Neumann boundary conditions $u'(0) = 0$, $u'(2) = 0$.

9.3.9. (*i*) Write the following differential equations in Sturm–Liouville form. (*ii*) If possible, write down a minimization principle that characterizes the solutions to the Dirichlet boundary value problem on the interval $[1,2]$. (a) $-e^x u'' - e^x u' = e^{2x}$, (b) $-x u'' - u' + 2u = 1$, (c) $-u'' - 2u' + u = e^x$, (d) $-x^2 u'' + 2xu' + 3u = 1$, (e) $xu'' + (1-x)u' + u = 0$.

9.3.10. *True or false*: The Sturm–Liouville operator (9.71) is self-adjoint and positive definite when subject to periodic boundary conditions $u(a) = u(b)$, $u'(a) = u'(b)$.

9.3.11. Does the quadratic functional $Q[u] = \int_0^1 \left[\frac{1}{2}(u')^2 - \left(x - \frac{1}{2}\right)u\right] dx$ have a minimum value when $u(x)$ is subject to the homogeneous Neumann boundary value conditions $u'(0) = u'(1) = 0$? If so, determine the minimum value and find all minimizing functions.

9.3.12. (a) Determine the adjoint of the differential operator $L[u] = u' + 2xu$ with respect to the L^2 inner products on $[0,1]$ when subject to the fixed boundary conditions $u(0) = u(1) = 0$. (*b*) Is the self-adjoint operator $S = L^* \circ L$ positive definite? Explain your answer. (*c*) Write out the boundary value problem represented by $S[u] = f$. (*d*) Find the solution to the boundary value problem when $f(x) = e^{x^2}$. *Hint*: To integrate the differential equation, work with the factored form of the differential operator. (*e*) Discuss what happens if you instead impose the Neumann boundary conditions $u'(0) = u'(1) = 0$.

9.3.13. Discuss the self-adjointness and positive definiteness of boundary value problems associated with the Bessel operator (9.79) of order $m = 0$.

9.3.14. Let $u_\star(x)$ be the solution to the self-adjoint positive definite boundary value problem $S[u_\star] = f$. Prove that if $f(x) \not\equiv 0$, then the minimum of the associated quadratic functional is strictly negative: $Q[u_\star] < 0$.

9.3.15. Find a function $u(x)$ such that $\int_0^1 u''(x)\, u(x)\, dx > 0$. How do you reconcile this with the claimed positivity in (9.70)?

9.3.16. Does the inequality (9.70) hold when $u(x) \not\equiv 0$ is subject to the Neumann boundary conditions $u'(a) = u'(b) = 0$?

9.3.17. *True or false*: When subject to homogeneous Dirichlet boundary conditions on an interval $[a, b]$, every nonsingular second-order linear ordinary differential equation $a(x)\, u'' + b(x)\, u' + c(x)\, u = f(x)$ is (a) self-adjoint, (b) positive definite, (c) positive semi-definite, with respect to some weighted inner product (9.76).

The Dirichlet Principle

9.3.18. (a) Show that the function $u(x, y) = \frac{1}{2}(-x\,y + x\,y^2 + x^2\,y - x^2\,y^2)$ solves the homogeneous Dirichlet boundary value problem for the Poisson equation $-\Delta u = x^2 + y^2 - x - y$ on the unit square $S = \{0 \le x \le 1,\ 0 \le y \le 1\}$. (b) Write down the Dirichlet integral (9.82) for this boundary value problem. What is its value for your solution? (c) Write down three other functions that satisfy the homogeneous Dirichlet boundary conditions on S, and check that all three have larger Dirichlet integrals.

9.3.19. (a) Suppose $u(x, y)$ solves the boundary value problem $-\Delta u = f$ in Ω and $u = 0$ on $\partial\Omega$, with $f(x, y) \not\equiv 0$. Prove that its Dirichlet integral (9.82) is strictly negative: $Q[u] < 0$. (b) Does this result hold for the inhomogeneous boundary value problem $u = h$ on $\partial\Omega$?

9.3.20. Consider the boundary value problem $-\Delta u = 1$, $x^2 + y^2 < 1$, $u = 0$, $x^2 + y^2 = 1$. (a) Find all solutions. (b) Formulate the Dirichlet minimization principle for this problem. Carefully indicate the function space over which you are minimizing. Make sure your solution belongs to the function space. (c) Which of the following functions belong to your function space? (i) $1 - x^2 - y^2$, (ii) $1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$, (iii) $x - x^3 - x\,y^2$, (iv) $x^4 - x^2\,y^2 + y^4$, (v) $\frac{1}{2}e^{-x^2-y^2} - \frac{1}{2}e^{-1}$. (d) For each function in part (c) that does belong to your function space, verify that its Dirichlet integral is larger than your solution's value.

9.3.21. Suppose $\lambda > 0$. Under what conditions does the inhomogeneous Neumann problem $-\Delta u + \lambda\, u = f$ in Ω, $\partial u/\partial \mathbf{n} = k$ on $\partial\Omega$, for the Helmholtz equation have a solution? Is the solution unique? *Hint*: Is the boundary value problem positive definite?

9.3.22. Suppose $\kappa(x) > 0$ for all $a \le x \le b$.

(a) Prove that the solution $u_\star(x)$ to the inhomogeneous Dirichlet boundary value problem

$$-\frac{d}{dx}\left(\kappa(x)\,\frac{du}{dx}\right) = f(x), \qquad u(a) = \alpha, \qquad u(b) = \beta,$$

minimizes the functional $Q[u] = \int_a^b \left[\,\frac{1}{2}\,\kappa(x)\,u'(x)^2 - f(x)\,u(x)\,\right] dx$.
Hint: Mimic the proof of Theorem 9.32.

(b) Construct a minimization principle for the mixed boundary value problem

$$-\frac{d}{dx}\left(\kappa(x)\,\frac{du}{dx}\right) = f(x), \qquad u(a) = \alpha, \qquad u'(b) = \beta.$$

9.3.23. Use the result of Exercise 9.3.22 to find the C^2 function $u_\star(x)$ that minimizes the integral $Q[u] = \int_1^2 \left[\frac{x}{2}\left(\frac{du}{dx}\right)^2 + x^2 u\right] dx$ when subject to the boundary conditions $u(1) = 0$, $u(2) = 1$.

9.3.24. Find the function $u(x)$ that minimizes the integral $Q[u] = \int_1^2 [\,x(u')^2 + x^2 u\,]\,dx$ subject to the boundary conditions $u(1) = 1$, $u'(2) = 0$. *Hint*: Use Exercise 9.3.22(b).

9.3.25. Prove that the functional $Q[u] = \int_0^1 (u')^2\,dx$, when subject to the mixed boundary conditions $u(0) = 0$, $u'(1) = 1$, has no minimizer.

9.3.26. Let $p_1(x,y), p_2(x,y), q(x,y) > 0$ be strictly positive functions on a closed, bounded, connected domain $\overline{\Omega} \subset \mathbb{R}^2$. Consider the boundary value problem for the second-order partial differential equation

$$-\frac{\partial}{\partial x}\left(p_1(x,y)\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(p_2(x,y)\,\frac{\partial u}{\partial y}\right) + q(x,y)\,u = f(x,y), \qquad (x,y) \in \Omega, \qquad (9.85)$$

subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$.
(a) *True or false*: Equation (9.85) is an elliptic partial differential equation. (b) Write the boundary value problem in self-adjoint form $L^* \circ L[u] = f$. *Hint*: Regard (9.85) as a "two-dimensional Sturm–Liouville equation". (c) Prove that this boundary value problem is positive definite, and then find a minimization principle that characterizes the solution. (d) Find suitable homogeneous Neumann–type boundary conditions involving the values of the derivatives of u on $\partial\Omega$ that make the resulting boundary value problem for (9.85) self-adjoint. Is your boundary value problem positive definite? Why or why not?

9.4 Eigenvalues and Eigenfunctions

9.4.1. Find the eigenvalues and an orthonormal eigenvector basis for the following symmetric matrices:
(a) $\begin{pmatrix} 2 & 6 \\ 6 & -7 \end{pmatrix}$, (b) $\begin{pmatrix} 5 & -2 \\ -2 & 5 \end{pmatrix}$, (c) $\begin{pmatrix} 2 & -1 \\ -1 & 5 \end{pmatrix}$, (d) $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 4 & 3 & 1 \end{pmatrix}$, (e) $\begin{pmatrix} 6 & -4 & 1 \\ -4 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix}$.

9.4.2. Determine whether the following symmetric matrices are positive definite by computing their eigenvalues.
(a) $\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -2 & 3 \\ 3 & 6 \end{pmatrix}$, (c) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$, (d) $\begin{pmatrix} 4 & -1 & -2 \\ -1 & 4 & -1 \\ -2 & -1 & 4 \end{pmatrix}$.

9.4.3. Suppose $S[\mathbf{u}] = K\mathbf{u}$, where $K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (a) Show that $S\colon \mathbb{R}^2 \to \mathbb{R}^2$ is positive semi-definite under the dot product. (b) Find the eigenvalues of S. (c) Explain why your result in part (b) does not contradict Theorem 9.34.

9.4.4. Suppose that $S\colon U \to U$ is a positive semi-definite linear operator. Let $\mathrm{I}\colon U \to U$ be the identity operator, so $\mathrm{I}[u] = u$. (a) Prove that, for any positive scalar $\mu > 0$, the operator $S_\mu = S + \mu \mathrm{I}$ is positive definite. (b) Show that S and S_μ have the same eigenfunctions. Do they have the same eigenvalues? If not, how are their eigenvalues related?

9.4.5. Find the minimum value of $R[v] = \dfrac{\int_0^1 v'^2\,dx}{\int_0^1 v^2\,dx}$ on the space of C^2 functions $v(x)$ defined on $0 \le x \le 1$ that are subject to one of the following pairs of boundary conditions:
(a) $v(0) = v(1) = 0$, (b) $v(0) = v'(1) = 0$, (c) $v'(0) = v'(1) = 0$.

9.4.6. Find the minimum value of $R[v] = \dfrac{\int_1^e x^2 v'^2\,dx}{\int_1^e v^2\,dx}$ on the space of C^2 functions defined on $[1, e]$ subject to the boundary conditions $v(1) = v(e) = 0$.

9.4.7. Show that the Rayleigh quotient $R[v]$ has the same value for all nonzero scalar multiples of an element $0 \ne v \in U$, i.e., $R[cv] = R[v]$ for all $c \ne 0$.

9.4.8. Prove that the minimum value of the Rayleigh quotient of a positive semi-definite, but not positive definite, operator is 0.

9.4.9. (a) Find the eigenfunctions and eigenvalues for the boundary value problem

$$-x^2u'' - xu' = \lambda\, u, \qquad u(1) = u(e) = 0.$$

(b) Under which inner product are the eigenfunctions orthogonal? Justify your answer by direct computation.
(c) Write down the eigenfunction expansion of a function $f(x)$ defined for $1 \le x \le e$.
(d) Find the Green's function for

$$-x^2u'' - xu' = f(x), \qquad u(1) = u(e) = 0,$$

both in closed form and as a series in the eigenfunctions you found in part (a).
(e) Is your Green's function symmetric? Discuss.
(f) Prove the completeness of the eigenfunctions.

9.4.10. Discuss completeness of the eigenfunctions of the boundary value problem

$$-x^2u'' - 2xu' = \lambda\, u, \qquad |\,u(0)\,| < \infty, \qquad u(1) = 0.$$

9.4.11. Consider the eigenvalue problem $-u'' = \lambda\, u$, $u(0) = 0$, $u'(1) = 0$. (a) Is the problem self-adjoint? positive definite? Which inner product are you referring to? (b) Find all eigenvalues and eigenfunctions. (c) Write down the explicit formula for the eigenfunction expansion of a function $f(x)$ defined on $[0,1]$. (d) Find the Green's function and use it to prove completeness of the eigenfunctions.

9.4.12. (a) Find the eigenfunctions and eigenvalues for the *Chebyshev boundary value problem*

$$(x^2-1)u'' + xu' = \lambda\, u, \qquad u(-1) = u(1) = 0.$$

Hint: Let $x = \cos\theta$. (b) Under what inner product are the eigenfunctions orthogonal? Justify your answer by direct computation. (c) Find the Green's function for

$$(x^2-1)u'' + xu' = f(x), \qquad u(-1) = u(1) = 0,$$

both in closed form and as a series in the eigenfunctions you found in part (a).
(d) Discuss completeness of the eigenfunctions.

9.4.13. Consider the differential operator $S[u] = -u'' + u$ on the space of C^2 functions $u(x)$ defined for all x and subject to the boundary conditions $\lim_{x\to\infty} u(x) = \lim_{x\to-\infty} u(x) = 0$.
(a) Find the Green's function $G(x;\xi)$. (b) Compute its double L^2 norm: $\|\,G\,\|^2$. What does this indicate about the completeness of the eigenfunctions of S? (c) Justify your conclusion in part (b) by determining the eigenfunctions.

9.4.14. Find all (real and complex) eigenvalues of the first-derivative operator $D = d/dx$ on the interval $[0,1]$ subject to the single periodic boundary condition $v(0) = v(1)$. Are the corresponding eigenfunctions orthogonal? For which inner product?

9.4.15. Consider the Dirichlet boundary value problem

$$-\Delta u = h(x,y), \quad u(x,0)=0, \quad u(x,1)=0, \quad u(0,y)=0, \quad u(1,y)=0, \quad 0<x,y<1,$$

for the Poisson equation on the unit square. (a) Find the eigenfunction series expansion for the Green's function of this problem. (b) Does your series coincide with that derived in Exercise 6.3.22? Explain any discrepancies. (c) For the impulse points $(\xi,\eta)=(.5,.5)$ and $(.7,.8)$, graph the result of summing the first 9, 25, and 100 terms in your series, and discuss what you observe in light of what you expect the Green's function to look like.

9.4.16. Find the eigenfunction series expansion for the Green's function of the following mixed boundary value problems:
(a) $-\Delta u = h(x,y), \quad u(x,0)=0, \quad u(x,1)=0, \quad u_x(0,y)=0, \quad u_x(1,y)=0, \quad 0<x,y<1;$
(b) $-\Delta u = h(x,y), \quad u(x,0)=0, \quad u_y(x,1)=0, \quad u(0,y)=0, \quad u_x(1,y)=0, \quad 0<x,y<1.$

9.4.17. Find the eigenfunction series expansion for the Green's function of the following Helmholtz boundary value problem:

$$-\Delta u + u = h(x,y), \qquad u(x,0)=u(x,\pi)=u(0,y)=u(\pi,y)=0, \qquad 0<x,y<\pi.$$

9.4.18. If the eigenvalues of a self-adjoint linear operator satisfy $\lambda_n \to \infty$ as $n \to \infty$, explain why each eigenspace is necessarily finite-dimensional.

9.4.19. *True or false*: If $S:\mathbb{R}^n \to \mathbb{R}^n$ is any linear function, then one can find an inner product on $\mathbb{R}^n$ that makes S self-adjoint.

9.5 A General Framework for Dynamics

9.5.1. Find the eigenfunction series of the fundamental solution for the heat equation $u_t = \gamma\, u_{xx}$ on the interval $0 \le x \le 1$ subject to homogeneous Dirichlet boundary conditions.

9.5.2. Solve Exercise 9.5.1 for (a) the mixed boundary conditions $u(t,0)=u_x(t,1)=0$; (b) homogeneous Neumann boundary conditions.

9.5.3. Let $D[u]=u'$ be the derivative operator acting on the vector space of C^1 scalar functions $u(x)$ defined for $0 \le x \le 1$ and satisfying the boundary conditions $u(0)=u'(1)=0$.
(a) Given the L^2 inner product on its domain space and the weighted inner product $\langle v, \tilde v\rangle = \int_0^1 v(x)\,\tilde v(x)\,x\,dx$ on its target space, determine the adjoint operator D^*.
(b) Let $S = D^* \circ D$. Write out the diffusion equation $u_t = -S[u]$ explicitly, as a partial differential equation plus boundary conditions.
(c) Given the initial condition $u(0,x)=x-x^2$, what is the asymptotic equilibrium $u_\star(x) = \lim_{t\to\infty} u(t,x)$ of the resulting solution to the diffusion equation?

9.5.4. Write down an eigenfunction series for the solution $u(t,x)$ to the initial value problem $u(0,x) = f(x)$ for the fourth-order evolution equation $u_t = -u_{xxxx}$ subject to the boundary conditions $u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0$. Does your solution tend to an equilibrium state? If so, at what rate?

9.5.5. Answer Exercise 9.5.4 for the boundary conditions

$$u_x(t,0) = u_{xxx}(t,0) = u_x(t,1) = u_{xxx}(t,1) = 0.$$

9.5.6. Explain how to solve the forced diffusion equation $u_t = -S[u] + f$, subject to homogeneous boundary conditions, when $f(x)$ does not depend on time t. Does the solution tend to equilibrium as $t \to \infty$? If so, what is the rate of decay, and what is the equilibrium?

9.5.7. Show that if $u(t,x)$ solves the diffusion equation (9.122), then $\| u(t,\cdot) \| \geq \| u(s,\cdot) \|$ whenever $t \leq s$.

9.5.8. Let $S > 0$ be a positive definite operator. Suppose $F(t,x;\xi)$ is the fundamental solution for the diffusion equation (9.122). Prove that $G(x;\xi) = \int_0^\infty F(t,x;\xi)\,dt$ is the Green's function for the corresponding equilibrium equation $S[u] = f$.

Vibration Equations

9.5.9. Which of the following forcing functions $F(t,x)$ excites resonance in the wave equation $u_{tt} = u_{xx} + F(t,x)$ when subject to homogeneous Dirichlet boundary conditions on the interval $0 \leq x \leq 1$? (a) $\sin 3t$, (b) $\sin 3\pi t$, (c) $\sin \frac{3}{2}\pi t$, (d) $\sin \pi t \, \sin \pi x$, (e) $\sin \pi t \, \sin 2\pi x$, (f) $\sin 2\pi t \, \cos \pi x$, (g) $x(1-x)\sin 2\pi t$.

9.5.10. Answer Exercise 9.5.9 when the solution is subject to the mixed boundary conditions $u(t,0) = u_x(t,1) = 0$.

9.5.11. Let $\omega > 0$. Find the solution to the initial-boundary value problem

$$u_{tt} = u_{xx} + \cos \omega t, \qquad u(t,0) = 0 = u(t,1), \qquad u(0,x) = 0 = u_t(0,x).$$

9.5.12. Answer Exercise 9.5.11 for homogeneous Neumann boundary conditions.

9.5.13. A piano wire of length 1 m and wave speed $c = 2$ m/sec can support a maximal deflection of 5 cm before breaking. Suppose the wire starts at rest, with both ends fixed, and then is subject to a uniform periodic force $F(t,x) = \frac{1}{10}\cos \omega t \, \sin \pi x$. What range of frequencies will cause the wire to break?

9.5.14. Write out the eigenfunction series solution to the initial-boundary value problem in Example 9.51 with $h(x) \equiv \sin k\pi x$.

9.5.15. How should the solution formulas (9.131, 134) be modified when there are unstable modes? Write down explicit conditions on the initial data that prevent an instability from being excited.

9.5.16. Explain how to convert the homogeneous wave equation with inhomogeneous Dirichlet boundary conditions $u(t,0) = \alpha(t)$, $u(t,\ell) = \beta(t)$, into a homogeneous boundary value problem for the forced wave equation. *Hint*: Mimic (4.46).

9.5.17. Two children hold a jump rope taut, while one of them periodically shakes their end of the rope. Use the inhomogeneous boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(t,0) = 0, \qquad u(t,1) = \sin \omega t,$$

to model the motion of the rope, adopting units in which the wave speed $c = 1$.
(*a*) What are the resonant frequencies of this system?
(*b*) Apply the method of Exercise 9.5.16 to find a particular solution to the boundary value problem when ω is a nonresonant frequency.
(*c*) Suppose the rope starts at rest. Find a series solution to the corresponding initial-boundary value problem when ω is a nonresonant frequency.
(*d*) Answer parts (*b,c*) when ω is a resonant frequency. *Hint*: Use the ansatz (9.143).

9.5.18. Explain how to solve the periodically forced *telegrapher's equation*

$$u_{tt} + a\,u_t = c^2\,u_{xx} + h(x)\cos\omega t$$

on the interval $0 \le x \le 1$ when subject to homogeneous Dirichlet boundary conditions. At which frequencies does the forcing function excite a resonant response?
Hint: First solve Exercise 4.2.9.

9.5.19. The fourth-order evolution equation $u_{tt} = -c^2 u_{xxxx}$, subject to the boundary conditions $u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0$, models the transverse vibrations of a simply supported uniform thin elastic beam, in which $c > 0$ represents the wave speed. Write down an eigenfunction series for the solution to the initial value problem $u(0,x) = f(x)$, $u_t(0,x) = 0$. Is the solution (*i*) periodic, (*ii*) quasiperiodic, (*iii*) chaotic, (*iv*) none of the above?

The Schroding Equation

9.5.20. (*a*) Solve the following initial boundary value problem:

$$\mathrm{i}\,\hbar\,\psi_t = -\,\psi_{xx}, \qquad \psi(t,0) = \psi(t,1) = 0, \qquad \psi(0,x) = 1.$$

(*b*) Using your solution formula, verify that $\|\,\psi(t,\cdot\,)\,\| = 1$ for all t.

9.5.21. Answer Exercise 9.5.20 for the initial condition $\psi(0,x) = \sqrt{30}\,x(1-x)$.

9.5.22. Answer Exercise 9.5.20 when the solution is subject to Neumann boundary conditions $\psi_x(t,0) = \psi_x(t,1) = 0$.

9.5.23. Write down the eigenseries solution for the Schrödinger equation on a bounded interval $[0, \ell]$ when subject to homogeneous Neumann boundary conditions.

9.5.24. Given the solution formula (9.152), and assuming completeness of the eigenfunctions, prove that $\| \psi(t, \cdot) \|^2 = \sum_k | c_k |^2$ for all t.

9.5.25. Write down the dispersion relation, phase velocity, and group velocity for the one-dimensional Schrödinger equation (9.153).

9.5.26. Show that the real and imaginary parts of the solution $\psi(t, x) = u(t, x) + \mathrm{i}\, v(t, x)$ to the one-dimensional Schrödinger equation (9.153) are solutions to the beam equation of Exercise 9.5.19. What is the wave speed?

9.5.27. *The Talbot effect for the linear Schrödinger equation*: Let $u(t, x)$ solve the periodic initial-boundary value problem

$$\mathrm{i}\, u_t = u_{xx}, \quad u(t, -\pi) = u(t, \pi), \quad u_x(t, -\pi) = u_x(t, \pi),$$

with initial data $u(0, x) = \sigma(x)$ given by the unit step function. Prove that when $t = \pi p/q$, where p, q are integers, the solution $u(t, x)$ is constant on each interval $\hbar \pi j/q < x < \hbar \pi (j + 1)/q$ for integers $j \in \mathbb{Z}$. *Hint*: Use Exercise 6.1.29(*d*).

9.5.28. The wave function $\psi(t, x)$ of a one-dimensional free quantum particle of mass m satisfies the Schrödinger equation $\mathrm{i}\, \psi_t = -\hbar\, \psi_{xx}/(2m)$ on the real line $-\infty < x < \infty$. Assuming that ψ and its x derivatives decay reasonably rapidly to zero as $| x | \to \infty$, prove that the particle's expected position $\langle x \rangle = \int_{-\infty}^{\infty} x\, | \psi(t, x) |^2\, dx$ moves on a straight line.
Hint: Prove that $\dfrac{d^2 \langle x \rangle}{dt^2} = 0$.