Solve

(a) $u_t + u^2 u_x = 0$ for $t \ge 0$，u(x,0)=1 if x<1，u(x,0)=0 if $x \ge 1$，where u is a weak solution of the equation

(b) $u_t + u^2 u_x = 0$ for $t \ge 0$，u(x,0)=0 if x<0，u(x,0)=1 if $0 \le x \le 1$，u(x,0)=0 if x>1，where u is the weak solution satisfying this entropy condition。

(a) Aweak solutionof a partial differential equation (PDE) is a generalization of the concept of a classical solution。While a classical solution requires the function to be sufficiently smooth (e.g., differentiable) and satisfy the PDE pointwise everywhere，a weak solution relaxes these requirements。Instead，it satisfies the PDE in an integral or distributional sense，making it applicable to problems where classical solutions may not exist due to discontinuities or singularities。

$u_t + u^2 u_x = 0$，$u(x,0) = \begin{cases} 1, x < 1 \\ 0, x \ge 1 \end{cases}$，where u is a weak solution of the equation。

This is a conservation law。

1. Idetify the flux function $u_t + (\frac{1}{3}u^3)_x = 0$，the flux function is $F(u) = \frac{1}{3}u^3$
2. Characteristics and shock formation
   - For $x < 1$, $u = 1$, so characteristics have speed $dx/dt = 1^2 = 1$.
   - For $x \ge 1$, $u = 0$, so characteristics are vertical ($dx/dt = 0$).
   - The discontinuity at $x = 1$ leads to intersecting characteristics, forming a shock.
3. Rankine-Hugoniot condition

   Shock speed $s = \dfrac{F(u_r) - F(u_l)}{u_r - u_l} = \dfrac{0 - \frac{1}{3}}{0 - 1} = \dfrac{1}{3}$
4. Weak solution

   (1) The shock propagates from x=1 with speed $\frac{1}{3}$

   (2) The solution is a step function with the shock at $x = 1 + \frac{1}{3}t$

$$u(x,t) = \begin{cases} 1, x < 1 + \frac{1}{3}t \\ 0, x \ge 1 + \frac{1}{3}t \end{cases}$$

(b) $u_t + u^2 u_x = 0$ for $t \geq 0$, $u(x,0) = \begin{cases} 0, x < 0 \\ 1, 0 \leq x \leq 1 \\ 0, x > 1 \end{cases}$, where u is the weak solution satisfying this entropy condition。

The weak solution of the PDE $u_t + u^2 u_x = 0$ with the given initial condition and entropy condition is constructed by considering a rarefaction wave at x=0 and a shock wave at x=1。

1. Rarefaction wave at x=0
    - The left discontinuity (from $u = 0$ to $u = 1$) forms a rarefaction wave because the characteristic speed increases ($F'(u) = u^2$).
    - The rarefaction solution is $u(x,t) = \sqrt{\frac{x}{t}}$ for $0 \leq x \leq t$.
2. Shock wave at x=1
    - The right discontinuity (from $u = 1$ to $u = 0$) forms a shock satisfying the Rankine-Hugoniot condition.
    - Shock speed: $s = \frac{1}{3}$, so the shock position is $x = 1 + \frac{t}{3}$.
3. Solution structure(valid for $t < \frac{3}{2}$)

$$u(x,t) = \begin{cases} 0, x < 0 \\ \sqrt{\frac{x}{t}}, 0 \leq x \leq t \\ 1, t < x < 1 + \frac{t}{3} \\ 0, x \geq 1 + \frac{t}{3} \end{cases}$$