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Ch8 Linear and Nonlinear Evolution Equations

8.1 The Fundamental Solution of the Heat Equation

8.1.1. Find the solution to the heat equation $u_t = u_{xx}$ on the real line having the following initial condition at time $t = 0$. Then sketch graphs of the resulting temperature distribution at times $t = 0, 1$, and 5.

(a) e^{-x^2}, (b) the step function $\sigma(x)$, (c) $e^{-|x|}$, (d) $\begin{cases} 1 - |x|, & |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$

8.1.2. On an infinite bar with unit thermal diffusivity, a concentrated unit heat source is instantaneously applied at the origin at time $t = 0$. A heat sensor measures the resulting temperature in the bar at position $x = 1$. Determine the maximum temperature measured by the sensor. At what time is the maximum achieved?

8.1.3. (a) Find the solution to the heat equation (8.6) whose initial data corresponds to a pair of unit heat sources placed at positions $x = \pm 1$. (b) Graph the solution at times $t = .1, .25, .5, 1$. (c) At what time(s) does the origin experience its maximum overall temperature? What is the maximum temperature at the origin?

8.1.4. (a) Use the Fourier transform to solve the initial value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad u(0,x) = \delta'(x-\xi), \qquad -\infty < x < \infty, \quad t > 0,$$

whose initial data is the derivative of the delta function at a fixed position ξ.

(b) Show that your solution can be written as the derivative $\partial F/\partial x$ of the fundamental solution $F(t,x;\xi)$. Explain why this observation should be valid.

8.1.5. Suppose that the initial data $u(0,x) = f(x)$ is real. Explain why the Fourier transform solution formula (8.13) defines a real function $u(t,x)$ for all $t > 0$.

8.1.6. (a) What is the maximum value of the fundamental solution at time t?

(b) Can you justify the claim that its width is proportional to $\sqrt{t}$?

8.1.7. Prove directly that (8.5) is indeed a solution to the heat equation, and, moreover, has the correct initial and boundary conditions.

8.1.8. Show, by a direct computation, that the final formula in (8.14) is a solution to the heat equation for all $t > 0$.

8.1.9. Justify formula (8.15).

8.1.10. According to Exercises 4.1.11–12, both the t and x partial derivatives of the fundamental solution solve the heat equation. (*a*) Write down the initial value problem satisfied by these two solutions. (*b*) Set $\xi = 0$ and then sketch graphs of each solution at several selected times. (*c*) Reconstruct each solution as a Fourier integral.

8.1.11. Let $u(t,x) = \dfrac{\partial F}{\partial x}(t,x;0)$ denote the x derivative of the fundamental solution (8.14).
(a) Prove that $u(t,x)$ is a solution to the heat equation $u_t = u_{xx}$ on the domain $\{-\infty < x < \infty,\ t > 0\}$. (*b*) For fixed x, prove that $\lim\limits_{t\to 0^+} u(t,x) = 0$. (*c*) Explain why, despite the results in parts (a) and (*b*), $u(t,x)$ is *not* a classical solution to the initial value problem $u_t = u_{xx}$, $u(0,x) = 0$. What is the classical solution? (*d*) What initial value problem does $u(t,x)$ satisfy?

8.1.12. Justify all the statements in Example 8.2.

8.1.13. (*a*) Solve the heat equation on an infinite bar when the initial temperature is equal to 1 for $|x| < 1$ and 0 elsewhere, while a unit heat source is applied to the same part of the bar $|x| < 1$ for a unit time period $0 < t < 1$. (*b*) At what time and what location is the bar the hottest? (*c*) What is the final equilibrium temperature of the bar?

8.1.14. An insulated bar 1 meter long, with constant diffusivity $\gamma = 1$, is taken from a freezer that is kept at $-10°$ C, and then has its ends kept at room temperature of $20°$ C. A soldering iron with temperature $350°$ C is continually held at the midpoint of the bar.
(*a*) Set up an initial value problem modeling the temperature distribution in the bar.
(*b*) Find the corresponding equilibrium temperature distribution.

8.1.15. Consider the heat equation with unit thermal diffusivity on the interval $0 < x < 1$ subject to homogeneous Dirichlet boundary conditions.
(*a*) Find a Fourier series representation for the fundamental solution $\widehat{F}(t,x;\xi)$ that solves the initial-boundary value problem
$$u_t = u_{xx}, \quad t > 0, \quad 0 < x < 1, \qquad u(0,x) = \delta(x-\xi), \quad u(t,0) = 0 = u(t,1).$$
Your solution should depend on t, x and the point ξ where the initial delta impulse is applied.
(*b*) For the value $\xi = .3$, use a computer program to sum the first few terms in the series and graph the result at times $t = .0001, .001, .01$, and $.1$. Make sure you have included enough terms to obtain a reasonably accurate graph.
(*c*) Compare your graphs with those of the fundamental solution $F(t,x;.3)$ on an infinite interval at the same times. What is the maximum deviation between the two solutions on the entire interval $0 \le x \le 1$?
(*d*) Use your fundamental solution $\widehat{F}(t,x;\xi)$ to construct a series solution to the general initial value problem $u(0,x) = f(x)$. Is your series the same as the usual Fourier series solution? If not, explain any discrepancy.

8.1.16. *True or false*: Periodic forcing of the heat equation at a particular frequency can produce resonance. Justify your answer.

8.1.17. Find the fundamental solution for the *cable equation* $v_t = \gamma v_{xx} - \alpha v$ on the real line. *Hint*: See Exercise 4.1.16.

8.1.18. The partial differential equation $u_t + c\,u_x = \gamma\,u_{xx}$ models transport of a diffusing pollutant in a fluid flow. Assuming that the speed c is constant, write down a solution to the initial value problem $u(0,x) = f(x)$ for $-\infty < x < \infty$. *Hint*: Look at Exercise 4.1.17.

8.1.19. Use the Fourier transform to solve the initial value problem $\mathrm{i}\,u_t = u_{xx}$, $u(0,x) = f(x)$, for the one-dimensional Schrödinger equation on the real line $-\infty < x < \infty$.

8.1.20. Let $u(t,x)$ be a solution to the heat equation having finite thermal energy, $E(t) = \int_{-\infty}^{\infty} u(t,x)\,dx < \infty$, and satisfying $u_x(t,x) \to 0$ as $x \to \pm\infty$, for all $t \geq 0$. Prove the law of *conservation of thermal energy*: $E(t) =$ constant.

8.1.21. Explain in your own words how a function $u(t,x)$ can satisfy $u(t,x) \to 0$ uniformly as $t \to \infty$ while maintaining the constancy of $\int_{-\infty}^{\infty} u(t,x)\,dx = 1$ for all t. Discuss what this signifies regarding the interchange of limits and integrals.

8.1.22. (a) Prove that if $\widehat{f}(k) \in \mathrm{L}^2$ is square-integrable, then so is $e^{-a k^2}\,\widehat{f}(k)$ for any $a > 0$. (b) Prove that when the initial data $f(x) \in \mathrm{L}^2$ is square integrable, so is the Fourier integral solution (8.13) for all $t \geq 0$.

8.1.23. Find the solution to the Black–Scholes equation for a put option (8.34).

8.1.24. (a) If we increase the interest rate r, does the value of a call option (*i*) increase; (*ii*) decrease; (*iii*) stay the same; (*iv*) could do any of the above? Justify your answer. (*b*) Answer the same question when rate stays fixed, but the volatility σ is increased.

8.1.25. Justify formula (8.42).

8.2 Symmetry and Similarity

8.2.1. If it takes a 2 cm long insulated bar 23 minutes to cool down to room temperature, how long does it take a 4 cm bar?

8.2.2. If it takes a 5 centimeter long insulated iron bar 10 minutes to cool down so as not to burn your hand, how long does it take a 20 centimeter bar made out of the same material to cool down to the same temperature?

8.2.3. (a) Given $\gamma > 0$, use a scaling transformation to write down the formula for the fundamental solution for the general heat equation $u_t = \gamma u_{xx}$ for $x \in \mathbb{R}$. (*b*) Write down the corresponding integral formula for the solution to the initial value problem.

8.2.4. Use scaling to construct the series solution for a heated circular ring of radius r and thermal diffusivity γ. Does scaling also give the correct formulas for the Fourier coefficients in terms of the initial temperature distribution?

8.2.5. A solution $u(t,x)$ to the heat equation is measured in degrees Fahrenheit. What is the corresponding temperature in degrees Kelvin? Which symmetry transformation takes the first solution to the second solution, and how does it affect the diffusion coefficient?

8.2.6. Is time reversal, $t \mapsto -t$, a symmetry of the heat equation? Write down a physical explanation, and then a mathematical justification.

8.2.7. According to Exercise 4.1.17, the partial differential equation $u_t + c\,u_x = \gamma\,u_{xx}$ models diffusion in a convective flow. Show how to use scaling to place the differential equation in the form $u_t + u_x = P^{-1} u_{xx}$, where P is called the *Péclet number*, and controls the rate of mixing. Is there a scaling that will reduce the problem to the case $P = 1$?

8.2.8. Suppose you know a solution $u^\star(t,x)$ to the heat equation that satisfies $u^\star(1,x) = f(x)$. Explain how to solve the initial value problem with $u(0,x) = f(x)$.

8.2.9. Solve the following initial value problems for the heat equation $u_t = u_{xx}$ for $x \in \mathbb{R}$:
(a) $u(0,x) = e^{-x^2/4}$. *Hint*: Use Exercise 8.2.8. (b) $u(0,x) = e^{-4x^2}$.
(c) $u(0,x) = x^2\,e^{-x^2/4}$. *Hint*: Use Exercise 4.1.12.

8.2.10. Define the functions $H_n(x)$ for $n = 0, 1, 2, \ldots$, by the formula

$$\frac{d^n}{dx^n}\,e^{-x^2} = (-1)^n H_n(x)\,e^{-x^2}. \tag{8.64}$$

(a) Prove that $H_n(x)$ is a polynomial of degree n, known as the n^{th} *Hermite polynomial*.
(b) Calculate the first four Hermite polynomials.
(c) Assuming $\gamma = 1$, find the solution to the heat equation for $-\infty < x < \infty$ and $t > 0$, given the initial data $u(0,x) = H_n(x)\,e^{-x^2}$. *Hint*: Combine Exercises 4.1.11, 8.2.8.

8.2.11. Find the scaling symmetries and corresponding similarity solutions of the following partial differential equations: (a) $u_t = x^2\,u_x$, (b) $u_t + u^2 u_x = 0$, (c) $u_{tt} = u_{xx}$.

8.2.12. Show that the wave equation $u_{tt} = c^2 u_{xx}$ has the following invariance properties: if $u(t,x)$ is a solution, so is (a) any time translate: $u(t-a,x)$, where a is fixed; (b) any space translate: $u(t,x-b)$, where b is fixed; (c) the dilated function $u(\beta t, \beta x)$ for $\beta \neq 0$; (d) any derivative: say $\partial u/\partial x$ or $\partial^2 u/\partial t^2$, provided u is sufficiently smooth.

8.2.13. Suppose $a = 0$, $b \neq 0$ in the scaling transformation (8.57).
(*a*) Discuss how to reduce the partial differential equation to an ordinary differential equation for the corresponding similarity solutions.
(*b*) Illustrate your method with the partial differential equation $t\,u_t = u\,u_{xx}$.

8.2.14. *True or false*: (*a*) A homogeneous polynomial solution to a partial differential equation is always a similarity solution. (*b*) An inhomogeneous polynomial solution to a partial differential equation can never be a similarity solution.

8.2.15. (*a*) Find all scaling symmetries of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$. (*b*) Write down the ordinary differential equation for the similarity solutions. (*c*) Can you find an explicit formula for the similarity solutions? *Hint*: Look at Exercise 8.2.14(a).

8.2.16. Besides the translations and scalings, Lie symmetry methods, [**87**], produce two other classes of symmetry transformations for the heat equation $u_t = u_{xx}$. Given that $u(t,x)$ is a solution to the heat equation:
(*a*) Prove that $U(t,x) = e^{c^2 t - cx}\, u(t, x - 2ct)$ is also a solution to the heat equation for any $c \in \mathbb{R}$. What solution do you obtain if $u(t,x) = a$ is a constant solution? *Remark*: This transformation can be interpreted as the effect of a *Galilean boost* to a coordinate frame that is moving with speed c.

(*b*) Prove that $U(t,x) = \dfrac{e^{-cx^2/(4(1+ct))}}{\sqrt{1+ct}}\, u\left(\dfrac{t}{1+ct}, \dfrac{x}{1+ct}\right)$ is a solution to the heat equation for any $c \in \mathbb{R}$. What solution do you obtain if $u(t,x) = a$ is a constant?

8.3 The Maximal Principle

8.3.1. *True or false*: Assuming no external heat source, if the initial and boundary temperatures of a one-dimensional body are always positive, the temperature within the body is necessarily positive.

8.3.2. Suppose $u(t,x)$ and $v(t,x)$ are two solutions to the heat equation such that $u \leq v$ when $t = 0$ and when $x = a$ or $x = b$. Prove that $u(t,x) \leq v(t,x)$ for all $a \leq x \leq b$ and all $t \geq 0$. Provide a physical interpretion of this result.

8.3.3. For $t > 0$, let $u(t,x)$ be a solution to the unforced heat equation on an interval $a < x < b$, subject to homogeneous Dirichlet boundary conditions. Prove that
$M(t) = \max\{\, u(t,x) \mid a \leq x \leq b \,\}$ is a nonincreasing function of t.

8.3.4. (*a*) State and prove a Maximum Principle for the *convection-diffusion equation*
$u_t = u_{xx} + u_x$. (*b*) Does the equation $u_t = u_{xx} - u_x$ also admit a Maximum Principle?

8.3.5. Consider the parabolic equation $\dfrac{\partial u}{\partial t} = x\,\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x}$ on the interval $1 < x < 2$, with initia and boundary conditions $u(0, x) = f(x),\ u(t, 1) = \alpha(t),\ u(t, 2) = \beta(t)$.
(a) State and prove a version of the Maximum Principle for this problem.
(b) Establish uniqueness of the solution to this initial-boundary value problem.

8.3.6. (a) Show that $u(t, x) = -x^2 - 2xt$ is a solution to the diffusion equation $u_t = x\,u_{xx}$.
(b) Explain why this differential equation does not admit a Maximum Principle.

8.3.7. Suppose that $u(t, x)$ is a nonconstant solution to the heat equation on the interval $0 < x < \ell$, with homogeneous (a) Dirichlet, (b) Neumann, or (c) mixed boundary conditions. Prove that the function $E(t) = \int_0^\ell u(t, x)^2\,dx$ is everywhere decreasing: $E(t_1) > E(t_2)$ whenever $t_1 < t_2$.

8.3.8. *True or false*: The wave equation $u_{tt} = c^2 u_{xx}$ satisfies a Maximum Principle. If true, clearly state the principle; if false, explain why not.

8.4 Nonlinear Diffusion

8.4.1. Find the solution to Burgers' equation that has the following initial data:

$$u(0, x) = \quad (a)\ \sigma(x), \qquad (b)\ \sigma(-x), \qquad (c)\ \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

8.4.2. Starting with the heat equation solution $v(t, x) = 1 + t^{-1/2}\, e^{-x^2/(4\gamma t)}$, find the corresponding solution to Burgers' equation and discuss its behavior.

8.4.3. Justify the solution formula (8.87).

8.4.4. (a) Prove that $\lim\limits_{z \to \infty} z\, e^{z^2} \operatorname{erfc} z = 1/\sqrt{\pi}$. (b) Show that when $a < b$, the Burgers' solution (8.87) converges to the rarefaction wave (2.54) in the inviscid limit $\gamma \to 0^+$.

8.4.5. *True or false*: If $u(t, x)$ solves Burgers' equation for the step function initial condition $u(0, x) = \sigma(x)$, then $v(t, x) = u_x(t, x)$ solves the initial value problem with $v(0, x) = \delta(x)$.

8.4.6. *True or false*: If $\widehat{v}(t,x)$ is as given in (8.84), then

$$\frac{\partial \widehat{v}}{\partial x} = \int_{-\infty}^{\infty} \frac{\xi - x}{2\gamma t}\, e^{-H(t,x;\xi)}\, d\xi,$$

and hence the solution to the Burgers' initial value problem (8.70–71) can be written as

$$u(t,x) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\xi}{t}\, e^{-H(t,x;\xi)}\, d\xi}{\displaystyle\int_{-\infty}^{\infty} e^{-H(t,x;\xi)}\, d\xi}, \qquad \text{where} \qquad H(t,x;\xi) = \frac{(x-\xi)^2}{4\gamma t} + \frac{1}{2\gamma}\int_0^{\xi} f(\eta)\, d\eta.$$

8.4.7. Show that if $u(t,x)$ solves Burgers' equation, then $U(t,x) = u(t,x-ct) + c$ is also a solution. What is the physical interpretation of this symmetry?

8.4.8. (*a*) What is the effect of a scaling transformation $(t,x,u) \longmapsto (\alpha t, \beta x, \lambda u)$ on Burgers' equation? (*b*) Use your result to solve the initial value problem for the rescaled Burgers' equation $U_t + \rho U U_x = \sigma U_{xx}$, $U(0,x) = F(x)$.

8.4.9. (*a*) Find all scaling symmetries of Burgers' equation. (*b*) Determine the ordinary differential equation satisfied by the similarity solutions. (*c*) *True or false*: The Hopf–Cole transformation maps similarity solutions of the heat equation to similarity solutions of Burgers' equation.

8.4.10. What happens if you nonlinearize the heat equation (8.75) using the change of variables

$$(a)\ \ v = \varphi^2; \qquad (b)\ \ v = \sqrt{\varphi}; \qquad (c)\ \ v = \log \varphi\,?$$

8.4.11. What partial differential equation results from applying the exponential change of variables (8.76) to:
(*a*) the wave equation $v_{tt} = c^2 v_{xx}$? (*b*) the Laplace equation $v_{xx} + v_{yy} = 0$?

8.5 Dispersion and Solitons

8.5.1. Sketch a picture of the solution for the initial value problem in Example 8.13 at times $t = -.1, -.5$, and -1.

8.5.2. (*a*) Write down an integral formula for the solution to the dispersive wave equation (8.90) with initial data $u(0,x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$ (*b*) Use a computer package to plot your solution at several times and discuss what you observe.

8.5.3. (*a*) Write down an integral formula for the solution to the initial value problem

$$u_t + u_x + u_{xxx} = 0, \qquad u(0,x) = f(x).$$

(*b*) Based on the results in Example 8.13, discuss the behavior of the solution to the initial value problem $u(0,x) = e^{-x^2}$ as t increases.

8.5.4. Find the (*i*) dispersion relation, (*ii*) phase velocity, and (*iii*) group velocity for the following partial differential equations. Which are dispersive? (*a*) $u_t + u_x + u_{xxx} = 0$, (*b*) $u_t = u_{xxxxx}$, (*c*) $u_t + u_x - u_{xxt} = 0$, (*d*) $u_{tt} = c^2 u_{xx}$, (*e*) $u_{tt} = u_{xx} - u_{xxxx}$.

8.5.5. Find all linear evolution equations for which the group velocity equals the phase velocity. Justify your answer.

8.5.6. Show that the phase velocity is greater than the group velocity if and only if the phase velocity is a decreasing function of k for $k > 0$ and an increasing function of k for $k < 0$. How would you observe this in a physical system?

8.5.7. (*a*) *Conservation of Mass*: Prove that $T = u$ is a density associated with a conservation law of the dispersive wave equation (8.90). What is the corresponding flux? Under what conditions is total mass conserved? (*b*) *Conservation of Energy*: Establish the same result for the energy density $T = u^2$. (*c*) Is u^3 the density of a conservation law?

8.5.8. Prove that when $t = \pi p/q$, where p, q are integers, the solution (8.102) is constant on each interval $\pi j/q < x < \pi(j+1)/q$ for integers $j \in \mathbb{Z}$. *Hint*: Use Exercise 6.1.29(*d*). *Remark*: The proof that the solution is continuous and fractal at irrational times is considerably more difficult, [**90**].

8.5.9. (*a*) Find the complex Fourier series representing the fundamental solution $F(t, x; \xi)$ to the periodic initial-boundary value problem (8.100). (*b*) Prove that at time $t = 2\pi p/q$, where p, q are relatively prime integers, $F(t, x; \xi)$ is a linear combination of delta functions based at the points $\xi + 2\pi j/q$. *Hint*: Use Exercise 6.1.29(*c*). (*c*) Let $u(t, x)$ be any solution to (8.100). Prove that $u(2\pi p/q, x)$ is a linear combination of a finite number of translates, $f(x - x_j)$, of the initial data.

The Korteweg-de Vries Equation

8.5.10. Justify the statement that the width of a soliton is proportional to the inverse of the square root of its speed.

8.5.11. Prove that the function (8.116) is a symmetric, monotone, exponentially decreasing function on either side of its maximum height of $3c$.

8.5.12. Let $u(t, x)$ solve the Korteweg–de Vries equation.
(*a*) Show that $U(t, x) = u(t, x - ct) + c$ is also a solution.
(*b*) Give a physical interpretation of this symmetry.

8.5.13. (*a*) Find all scaling symmetries of the Korteweg–de Vries equation.
(*b*) Write down an ansatz for the similarity solutions, and then find the corresponding reduced ordinary differential equation. (Unfortunately, the similarity solutions cannot be written in terms of elementary functions, [**2**].)

8.5.14. (*a*) Let $u(t,x)$ be the two-soliton solution defined in (8.118). Let $\widetilde{u}(t,\xi) = u(t,\xi+ct)$ represent the solution as viewed in a coordinate frame moving with speed c. Prove that

$$\lim_{t\to\infty} \widetilde{u}(t,\xi) = \begin{cases} 3c_1 \operatorname{sech}^2\left[\frac{1}{2}\sqrt{c_1}\,\xi + \delta_1\right], & c = c_1, \\ 3c_2 \operatorname{sech}^2\left[\frac{1}{2}\sqrt{c_2}\,\xi + \delta_2\right], & c = c_2, \\ 0, & \text{otherwise}, \end{cases}$$

for suitable constants δ_1, δ_2. Explain why this justifies the statement that the solution indeed breaks up into two individual solitons as $t \to \infty$. (*b*) Explain why $\widetilde{u}(t,\xi)$ has a similar limiting behavior as $t \to -\infty$, but with possibly different constants $\widehat{\delta}_1, \widehat{\delta}_2$. (*c*) Use your formulas to discuss how the solitons are affected by the collision.

8.5.15. Let $\alpha, \beta \neq 0$. Find the soliton solutions to the rescaled Korteweg–de Vries equation $u_t + \alpha u_{xxx} + \beta u u_x = 0$. How are their speed, amplitude, and width interrelated?

8.5.16. (*a*) Find the solitary wave solutions to the *modified Korteweg–de Vries equation* $u_t + u_{xxx} + u^2 u_x = 0$. (*b*) Discuss how the amplitude and width of the solitary waves are related to their speeeds. *Note*: The modified Korteweg–de Vries equation is also integrable, and its solitary wave solutions are solitons, cf. [**36**].

8.5.17. Answer Exercise 8.5.16 for the *Benjamin–Bona–Mahony equation* $u_t - u_{xxt} + u u_x = 0$, [**14**]. *Note*: The BBM equation is *not* integrable, and collisions between its solitary waves produce a small, but measurable, inelastic effect, [**1**].

8.5.18. (*a*) Show that $T_1 = u$ is the density for a conservation law for the Korteweg–de Vries equation. (*b*) Show that $T_2 = u^2$ is also a conserved density. (*c*) Find a conserved density of the form $T_3 = u_x^2 + \mu u^3$ for a suitable constant μ. *Remark*: The Korteweg–de Vries equation in fact has *infinitely many* conservation laws, whose densities depend on higher and higher-order derivatives of the solution, [**76, 87**]. It was this discovery that unlocked the door to all its remarkable integrability properties, [**2, 36**].

8.5.19. Find two conservation laws of
(*a*) the modified Korteweg–de Vries equation $u_t + u_{xxx} + u^2 u_x = 0$;
(*b*) the Benjamin–Bona–Mahony equation $u_t - u_{xxt} + u u_x = 0$.