§ 調和函數習作

1. True or false：
   (1) Every harmonic polynomial is homogeneous。
   (2) very homogeneous polynomial is harmonic。
2. Show that a function which is a power series in the complex variable x+iy must satisfy the Cauchy－Riemann equations and therefore Laplace equation。
3. Solve the equation $u_x{}^2 + u_y{}^2 = e^{2y}$，u(0,y)=0
4. Solve $u_{xx} + u_{yy} = 1$ in r<a with u(x,y) vanishing on r=a。
5. Solve $u_{xx} + u_{yy} = 1$ in the annulus(圓環) a<r<b with u(x,y) vanishing on both parts of the boundary r=a an r=b。
6. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell a<r<b with u(x,y,z) vanishng on both the inner and outer boundaries。
7. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell a<r<b with u=0 on r=a and $\frac{\partial u}{\partial r} = 0$ on r=b。Then let $a \to 0$ in your answer and interpret the result。
8. A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution。Its inner boundary is held at $100^0 C$。Its outer boundary satisfies $\frac{\partial u}{\partial r} = -\gamma < 0$，where γ is a constant。
   (a) Find the temperature。(Hint：the temperature depends only on the radius。)
   (b) What are the hottest and coldest temperatures？
   (c) Can you choose γ so that the temperature on its outer boundary is $20^0 C$？
9. Check the validity of the maximum principle for the harmonic function $u(x,y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2}$ in the disk $\overline{D} = \{x^2 + y^2 \le 1\}$
10. Solve $u_{xx} + u_{yy} = 0$ in the rectangle 0<x<a，0<y<b with the following boundary conditions：$u_x = -a$ on x=0，$u_x = 0$ on x=a；$u_y = b$ on y=0，$u_y = 0$ on y=b
11. Find the harmonic function u(x,y) in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions：$u_y = 0$ for y=0 and for $y = \pi$，u=0 for x=0 and

$u=\cos^2 y=\frac{1}{2}(1+\cos 2y)$ for $x=\pi$

12. Find the harmonic function in the square {0<x<1，0<y<1}with the boundary conditions $u(x,0)=x, u(x,1)=0, u_x(0,y)=0, u_x(1,y)=y^2$

13. Solve the following Neumann problem in the cube {0<x<1，0<y<1，0<z<1}：

    $\Delta u=0$ with $u_z(x,y,1)=g(x,y)$ and homogeneous Neumann conditions on the other five faces，where g(x,y) is an arbitrary function with zero average。

14. Find the harmonic function in the semi-infinite strip $\{0\le x\le \pi, 0\le y<\infty\}$ that satisfies the boundary condition：

    $u(0,y)=u(\pi,y)=0, u(x,0)=h(x), \lim_{y\to\infty} u(x,y)=0$

15. Suppose that u is a harmonic function in the disk D={r<2} and that $u=3\sin 2\theta+1$ for r=2。Without finding the solution，answer the following questions

    (a) Find the maximum value of u in $\overline{D}$

    (b) Calculus the value of u at the origin。

16. Solve $u_{xx}+u_{yy}=0$ in the disk {r<a} with the boundary condition $u=1+3\sin\theta$ on r=a

17. Solve $u_{xx}+u_{yy}=0$ in the disk {r<a} with the boundary condition $u=\sin^3\theta$

18. Solve $u_{xx}+u_{yy}=0$ in the the exterior {r > a} of a disk，with the boundary condition $u=1+3\sin\theta$ on r=a，and the condition at infinity that u be bounded as $r\to\infty$

19. Solve $u_{xx}+u_{yy}=0$ in the disk r<a with the boundary condition $\frac{\partial u}{\partial r}-hu=f(\theta)$，where $f(\theta)$ is an arbitary function。Write the answer in terms of the Fourier coefficients of $f(\theta)$ 。

20. Find the steady-state temperature distribution inside an annular plate {1<r<2}，whose outer edge (r=2) is insulated，and on whose inner edge (r=1) the temperature is maintained as $\sin^2\theta$ 。

21. Find the harmonic function u in the semidisk $\{r<1, 0<\theta<\pi\}$ with u vanishing on the diameter $(\theta=0,\pi)$ and $u=\pi\sin\theta-\sin 2\theta$ on r=1

22.

解答

1. (1) 反例 $f(x,y)=x^2-y^2+x$ (2) 反例 $f(x,y)=x^2$

2. z=x+iy，f(z)=u(z)+iv(z)=u(x,y)+iv(x,y)， $f(z)=\sum_{n=0}^{\infty}a_n z^n$

   因為冪級數在收斂區域內逐項可微，因此對 x 和 y 微分：

   $$\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\sum_{n=1}^{\infty}a_n n(x+iy)^{n-1},\frac{\partial f}{\partial y}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}=i\sum_{n=1}^{\infty}a_n n(x+iy)^{n-1}$$

   $i(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x})=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$ ，可以得到 Cauchy-Riamann equations：

   $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ 。

   $u_{xx}=v_{yx}=v_{xy}=-u_{yy}$ then $\Delta u=0$

   同理 $\Delta v=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=-(\frac{\partial^2 u}{\partial y\partial x})+\frac{\partial^2 u}{\partial x\partial y}=0$

3. Assume a solution of the form $u(x,y)=e^y v(x)$

   $u_x=e^y v'(x), u_y=e^y v(x)$

   $v'^2+v^2=1$ This ODE has solutions v(x)=sin(x+c) or v(x)=cos(x+c)

   $u(x,y)=e^y\sin x$

4. $\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr})=1$ with u(a)=0 $u(r)=\frac{1}{4}(r^2-a^2)$

5. $\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr})=1\Rightarrow u(r)=\frac{1}{4}r^2+c\ln r+d$ with u(a)=u(b)=0

   $$u(r)=\frac{1}{4}\{r^2-\frac{b^2\ln(\frac{r}{a})+a^2\ln(\frac{b}{r})}{\ln(\frac{b}{a})}\}$$

6. $\nabla^2 u=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})=1$

   $u(r)=\frac{1}{6}r^2-\frac{c_1}{r}+c_2$ ，set u(a)=u(b)=0 解 $c_1=\frac{-ab(a+b)}{6},c_2=-\frac{a^2+ab+b^2}{6}$

   $u(r)=\frac{1}{6}(r^2-\frac{ab(a+b)}{r}-a^2-ab-b^2)$

7. $u(r)=\frac{r^2-a^2}{6}+\frac{b^3}{3}(\frac{1}{r}-\frac{1}{a})$

8. (a)Steady-state means the temperature has stabilized and remains constant over time at

every point in the shell。

The steady-state temperature distribution within the spherical shell is determined by solving Laplace's equation in spherical coordinates with radial symmetry。

$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr})=0$ ， $u(r)=\frac{A}{r}+B$

r=1，u(1)=100，at r=2， $\frac{\partial u}{\partial r}=-\gamma<0\Rightarrow A=4\gamma$

$u(r)=\frac{4\gamma}{r}+100-4\gamma$

(b) $\because u(r)$ is decreasing。The hottest temperature is u(1)=100^0C ，the coldest

temperature is $u(2)=100-2\gamma^0C$

(c) $\gamma=40$ at r=2

9. u(x,y) is singular at (1,0)，where it becomes discontinuous。
The maximum principle requires harmonicity in the open domain and continuity on the closure。

Since u fails to be continuous on the closed disk $\overline{D}$，the maximum principle does not apply。

10. $u(x,y)=\frac{1}{2}x^2-ax-\frac{1}{2}y^2+by+c$

11. Assume $u(x,y)=X(x)Y(y)$

$\nabla^2u=0\Rightarrow\begin{cases}X''-\lambda X=0\\Y''+\lambda Y=0\end{cases}$

Boundary condition： $Y'(0)=Y'(\pi)=0$

$Y''+\lambda Y=0\Rightarrow Y=A\sin\sqrt{\lambda}y+B\cos\sqrt{\lambda}y$

$Y'(0)=Y'(\pi)=0\Rightarrow\lambda_n=n^2,Y_n(y)=\cos(ny),n=0,1,2,...$

$X''-n^2X=0\Rightarrow X_n=Ae^{nx}+Be^{-nx}$ ，u(0,y)=0$\Rightarrow A+B=0$

Hence $X_n=C_n\sinh(nx)$ (except n=0)

$u(x,y)=\frac{A_0x}{2}+\sum_{n=1}^{\infty}C_n\sinh(nx)\cos(ny)$

Apply boundary condition at $x=\pi$， $\frac{A_0\pi}{2}+\sum_{n=1}^{\infty}C_n\sinh(n\pi)\cos(ny)=\frac{1}{2}(1+\cos 2y)$

$$u(x,y)=\frac{x}{2\pi}+\frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}$$

$$\Rightarrow A_0=\frac{1}{\pi},C_2=\frac{1}{2\sinh(2\pi)}$$

12. Assume u(x,y)=v(x,y)+w(x,y) ，v(x,0)=x,v(x,1)=0,v(0,y)=v(1,y)=0，

    $w_x(0,y)=0,w_x(1,y)=y^2$ ，w(x,0)=w(x,1)=0

    分別解 v，w 然後相加。

    其中用變數分離法解 v(x,y)，得 $v(x,y)=x-\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sinh(n\pi y)$

    $w(x,y)=\sum_{n=1}^{\infty}C_n\cosh(n\pi x)\sinh(n\pi y)$ ，其中 $C_n=\frac{2(-1)^n}{\pi^3 n^3\sinh(n\pi)}$
13. ⋯
14. ⋯
15. $u(r,\theta)=1+\frac{3r^2}{4}\sin 2\theta$
16. $u(r,\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}r^n(A_n\cos(n\theta)+B_n\sin(n\theta))$

    邊界條件為 $u(a,\theta)=1+3\sin\theta$ ，展開傅立葉級數

    $1+3\sin\theta=A_0+\sum_{n=1}^{\infty}a^n(A_n\cos n\theta+B_n\sin n\theta)\cdots$

    $u(r,\theta)=1+\frac{3r}{a}\sin\theta$
17. $u(r,\theta)=(\frac{3r}{4a})\sin\theta-(\frac{r^3}{4a^3})\sin 3\theta$
18.