§ 6.1 Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$

或者寫成 $\Delta u = \nabla \cdot \nabla u$ (subharmonic $\Leftrightarrow \Delta u \geq 0$)

A solution of the Laplace equation is called a harmonic function。
$\Delta u = f$ with a given function is called Poisson equation。

Δ (the **Laplace operator**) is defined as： $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$

Polar coordinates (r, θ)：

$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Shperical coordinates (r, θ, ϕ)： $\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \frac{\partial u}{\partial r}) + \ldots$

Laplace-Beltrami operator Δ_g is a generalization of the Laplace operator to functions defined on Riemannian manifolds。

$$f : M \to R \quad \Delta_g f = div_g(grad_g f) = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f)$$

in local coordinates $(x^1, \ldots, x^n)$

$f : \Omega \subset R^n \to R$

Key properties：

1. Mean value property

   For any ball $B(x,r) \subset \Omega$ ， $u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B} u(y) dS(y)$

2. Maximum principle
3. Smoothness
4. Liouville theorem

   A bounded harmonic function on R^n must be constant。

   There is no non-constant negative harmonic function defined on the Euclidean space。

   There is no non-constant negative subharmonic function on R^2 。

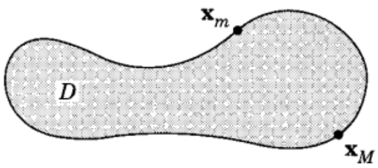
Examples：

1. The function $u(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}}$ is harmonic everywhere except at the origin。
2. Electrostatics(靜電學)
   curlE=0， $divE=4\pi\rho$
   For the electric potential ϕ ， $\Delta\phi = div(grad\phi) = -div$E=-4$\pi\rho$
3. Steady fluid flow
4. Analytic functions of a complex variable Cauchy-Riemann equation
   z=x+iy，f(z)=u(z)+iv(z) is an analytic function if $f(z)=\sum_{n=0}^{\infty} a_n z^n$
5. Brownian motion (或稱為 Wiener process)

布朗運動可以用偏微分方程來描述，其核心是熱方程，也稱為擴散方程。
在數學上，布朗運動 Bt 是滿足以下隨機微分方程（SDE）的隨機過程：
$dB_t = \sigma dW_t$
其中W_t是標準維納過程（Wiener process），σ是擴散係數。
布朗運動的 Fokker-Planck 方程：$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}$
其中 p(x,t)是機率密度函數。
對於自由布朗運動（無外力），該 PDE 的解為高斯分布：
$$p(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp(-\frac{x^2}{2\sigma^2 t})$$
The Feynman-Kac formula relates solutions of certain PDEs to expectations of stochastic processes involving Brownian motion。
For example，the solution to the PDE：
$$\frac{\partial u}{t} + \mu(x)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2} = 0$$
with terminal condition $u(T,x)=\phi(x)$，can be expressed as：
$$u(t,x) = E[\phi(X_T) | X_t = x]$$
where X_t is a stochastic process driven by Brownian motion。
In summary，Brownian motion is a stochastic process that bridges PDEs and probability theory，providing a probabilistic interpretation of solutions to certain PDEs and enabling the modeling of random phenomena。

§ maximum principle

Let D be a conncted bounded open set。Let u(x,y,z) be a harmonic function in D that is continuous on $\overline{D}$ $(=D\cup\partial D)$。Then the maximum and the minimum values of u are



attained on ∂D and nowhere inside。(unless $u\equiv$conatsnt)

> 有朋自遠方來 訪問 Robert Finn 提到 Eberhard Hopf 的 strong maximum principle。
> 恰好看到此章提到 maximum principle。
> [maximum principle $u_t = ku_{xx}$ is a one-dimensional **diffusion equation** PDE102-2]
> [高微 Extreme of functions of two variables]

§ rotational invariance

The Laplace equation is invariant under all rigid motions。

In engineering the Laplacian is a model for isotropic physical situations，in which there is no preferred direction。

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} \text{，} \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \quad \cdots(*)$$

$x = r\cos\theta, y = r\sin\theta$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}$$

$r = \sqrt{x^2+y^2}, \theta = \tan^{-1}\frac{y}{x}$ to find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}$

例如 $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} \text{，} \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial^2}{\partial x^2}=(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta})(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta})=\cos^2\theta\frac{\partial^2}{\partial r^2}+\cos\theta\frac{\partial}{\partial r}(-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta})+\ldots$$

$$\frac{\partial^2}{\partial x^2}=\cos^2\theta\frac{\partial^2}{\partial r^2}+\frac{\sin^2\theta}{r}\frac{\partial}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}.$$

$$\frac{\partial^2}{\partial y^2}=\sin^2\theta\frac{\partial^2}{\partial r^2}+\frac{\cos^2\theta}{r}\frac{\partial}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}.$$

這裡有很複雜的計算

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

若 harmonic functions 本身是旋轉不變，則(*)變成 $0=u_{rr}+\frac{1}{r}u_r$ ，若 u 與 θ 無關，此方程變成 $(ru_r)_r=0$ ， $u=c_1\ln r+c_2$

後面證明 3 維 Laplacian 在空間剛體運動下皆為不變量(暫略)。

Exercises

1. Show that a function which is a power series in the complex variable x+iy must satisfy the Cauchy－Riemann equations and therefore Laplace equation。

z=x+iy，f(z)=u(z)+iv(z)=u(x,y)+iv(x,y) $f(z)=\sum_{n=0}^{\infty}a_n z^n$

因為冪級數在收斂區域內逐項可微，因此對 x 和 y 微分：

$$\frac{\partial f}{\partial x}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\sum_{n=1}^{\infty}a_n n(x+iy)^{n-1},\frac{\partial f}{\partial y}=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}=i\sum_{n=1}^{\infty}a_n n(x+iy)^{n-1}$$

$i(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x})=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}$ ，可以得到 Cauchy-Riamann equations：

$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ 。

$u_{xx}=v_{yx}=v_{xy}=-u_{yy}$ then $\Delta u=0$

同理 $\Delta v=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=-(\frac{\partial^2 u}{\partial y\partial x})+\frac{\partial^2 u}{\partial x\partial y}=0$

2. Find the solutions that depend only on r of the equation $u_{xx}+u_{yy}+u_{zz}=k^2u$ ，where k is a positive constant。

球坐標系 (r,θ,ϕ) 下， $x=r\sin\theta\cos\phi, y=r\sin\theta\sin\phi, z=r\cos\theta$

Laplacian 的表式為：

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

與θ無關時，Laplacian 用球面座標表示為$\nabla^2 u = \frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr})$

$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr}) = k^2 u \Rightarrow \frac{d}{dr}(r^2\frac{du}{dr}) = k^2 r^2 u$ ，let v(r)=ru(r)， $u = \frac{v}{r} \Rightarrow \frac{du}{dr} = \frac{v'}{r} - \frac{v}{r^2}$

$r^2\frac{du}{dr} = rv' - v$ ， $\frac{d}{dr}(r^2\frac{du}{dr}) = \frac{d}{dr}(rv' - v) = v' + rv'' - v' = rv''$

$rv'' = k^2 r^2 u = k^2 r^2 \times \frac{v}{r} = k^2 rv \quad \Rightarrow v'' = k^2 v$

$v(r) = Ae^{kr} + Be^{-kr}$

$u = \frac{Ae^{kr} + Be^{-kr}}{r}$

3. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} = k^2 u$ ，where k is a positive constant。

The given equation is the Helmholtz equation。

$\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) = k^2 u \Rightarrow u_{rr} + \frac{1}{r}u_r = k^2 u \cdots(1)$

Let s=kr，(1)兩邊同乘以r^2 ，the equation becomes：

$s^2\frac{d^2u}{ds^2} + s\frac{du}{ds} - s^2 u = 0$ (a modified Bessel differential equation)

$u(r) = c_1 I_0(kr) + c_2 K_0(kr)$

where I_0 and K_0 are the modified Bessel functions of the first and second kind，respectively，and c_1, c_2 are constants。

The Besseel differential equation：

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0$

General solution is $y(x) = c_1 J_v(x) + c_2 Y_v(x)$

The modified Bessel differential equation：

4. Solve $u_{xx}+u_{yy}+u_{zz}=0$ in the spherical shell 0<a<r<b with the boundary condition u=A on r=a and u=B on r=b，where A and B are constants。

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr})=0\Rightarrow u=-\frac{c_1}{r}+c_2\Rightarrow u=\frac{C_1}{r}+c_2$$

At r=A，$A=\frac{C_1}{a}+c_2$；at r=B，$B=\frac{C_1}{b}+c_2$ 解出C_1,c_2

$$u(r)=\frac{Aa(b-r)+Bb(r-a)}{r(b-a)}$$

5. Solve $u_{xx}+u_{yy}=1$ in r<a with u(x,y) vanishing on r=a。

$\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr})=1$ with u(a)=0

$$u(r)=\frac{1}{4}(r^2-a^2)$$

6. Solve $u_{xx}+u_{yy}=1$ in the annulus(圓環) a<r<b with u(x,y) vanishing on both parts of the boundary r=a an r=b。

$\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr})=1\Rightarrow u(r)=\frac{1}{4}r^2+c\ln r+d$ with u(a)=u(b)=0

$$u(r)=\frac{1}{4}\{r^2-\frac{b^2\ln(\frac{r}{a})+a^2\ln(\frac{b}{r})}{\ln(\frac{b}{a})}\}$$

7. Solve $u_{xx}+u_{yy}+u_{zz}=1$ in the spherical shell a<r<b with u(x,y,z) vanishng on both the inner and ouer boundaries。

$$\nabla^2 u=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})=1$$

$u(r)=\frac{1}{6}r^2-\frac{c_1}{r}+c_2$，set u(a)=u(b)=0 解$c_1=\frac{-ab(a+b)}{6},c_2=-\frac{a^2+ab+b^2}{6}$

$$u(r)=\frac{1}{6}(r^2-\frac{ab(a+b)}{r}-a^2-ab-b^2)$$

8. Solve $u_{xx}+u_{yy}+u_{zz}=1$ in the spherical shell a<r<b with u=0 on r=a and $\frac{\partial u}{\partial r}=0$ on r=b。Then let $a\to 0$ in your answer and interpret the result。

$$u(r)=\frac{r^2-a^2}{6}+\frac{b^3}{3}(\frac{1}{r}-\frac{1}{a})$$

Taking the limit $a\to 0$，the term $-\frac{b^3}{3a}$ becomes singular。This indicates the solution develops a singularity at the origin，corresponding to an implicit Dirac delta source。

…這裡說明不懂。

9. A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution。Its inner boundary is held at 100^0C。Its outer boundary satisfies $\frac{\partial u}{\partial r}=-\gamma<0$，where γ is a constant。
   (a) Find the temperature。(Hint：the temperature depends only on the radius。)
   (b) What are the hottest and coldest temperatures？
   (c) Can you choose γ so that the temperature on its outer boundary is 20^0C？

(a) Steady-state means the temperature has stabilized and remains constant over time at every point in the shell。

The steady-state temperature distribution within the spherical shell is determined by solving Laplace's equation in spherical coordinates with radial symmetry。

$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr})=0$，$u(r)=\frac{A}{r}+B$

r=1，u(1)=100，at r=2，$\frac{\partial u}{\partial r}=-\gamma<0\Rightarrow A=4\gamma$

$$u(r)=\frac{4\gamma}{r}+100-4\gamma$$

(b) $\because u(r)$ is decreasing。The hottest temperature is u(1)=100^0C，the coldest temperature is $u(2)=100-2\gamma^0C$

(c) $\gamma=40$ at r=2

10. Prove the uniqueness of the Dirichlet problem $\Delta u=f$ in D，with u=g on bdyD by the energy method。That is，after subtracting two solutions w=u-v，multiply the Laplace equation for w by w itself and use the divergence theorem。
    (1) Assume two solutions u_1,u_2 and define $w=u_1-u_2$
    Since $\Delta u_1=\Delta u_2=f$，we have $\Delta w=0$ in D
    On the boundary，$w=u_1-u_2=g-g=0$
    (2) Apply Green first identity

$\int_D w\Delta w dx = 0$

$\int_D |\nabla w|^2 dx = \int_{\partial D} w\frac{\partial w}{\partial n} dS - \int_D w\Delta w dx = 0$

$|\nabla w|^2 \geq 0 \Rightarrow \nabla w \equiv 0 \Rightarrow w$ is constant，but w=0 on ∂D，implies the constant is zero。Therefore $u_1 = u_2$

Green theorem：

$$\int_C Mdx + Ndy = \iint_R (\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})dxdy$$

Divergence theorem：

$$\iint_S \vec{E}\cdot\vec{n}dS = \iiint_V div\vec{E}dV$$ ...Gauss 定理(散度定理)

Green's first identity: $\int_D (\nabla\phi\cdot\nabla\psi + \phi\Delta\psi)\,dV = \int_{\partial D} \phi\frac{\partial\psi}{\partial n}dS$

1. **Green's First Identity** (used to derive the second identity):

$$\iiint_D \nabla u\cdot\nabla v\,dV = \iint_{\partial D} u\frac{\partial v}{\partial n}dS - \iiint_D u\Delta v\,dV.$$

2. **Second Identity**: Subtract the first identity for u and v swapped:

$$\iint_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dS = \iiint_D (u\Delta v - v\Delta u)\,dV.$$

Green's first identity is a fundamental result in vector calculus that relates volume integrals over a domain D to surface integrals over its boundary ∂D。It is derived from the divergence theorem and serves as a higher-dimensional analog of integration by parts。

Formally, for two sufficiently smooth scalar functions ϕ and ψ defined on a domain $D \subset \mathbb{R}^n$ with boundary ∂D, Green's first identity states:

$$\int_D (\nabla\phi\cdot\nabla\psi)\,dV + \int_D \phi\,\Delta\psi\,dV = \int_{\partial D} \phi\frac{\partial\psi}{\partial n}dS,$$

Where $\nabla\phi\cdot\nabla\psi$ is the dot product of the gradients of $\phi$and ϕ 。

$\frac{\partial\psi}{\partial n} = \nabla\psi \bullet n$ is the normal derivative of ψ on ∂D。(n is the outward unit normal vector to ∂D)。

11. Show that there is no solution of $\Delta u = f$ in D，$\frac{\partial u}{\partial n} = g$ on bdyD in three dimensions，unless $\iiint_D f dxdydz = \iint_{\partial D} g dS$ 。Also show the analogue in one and two dimensions。

To demonstrate the necessity of the compatibility condition for the existence of a solution to the Neumann problem $\Delta u = f$ in D with $\frac{\partial u}{\partial n} = g$ on ∂D，we proceed as follows：

1. Integrate both sides of the Poisson equation over the domain D
   $\iiint_D \Delta u dV = \iiint_D f dV$
2. Apply the divergence theorem to the left-hand side $\iiint_D \nabla \cdot (\nabla u) dV = \iint_{\partial D} \nabla u \cdot n dS$
   Where n is the outward unit normal。
   Substituting the Neumann boundary condition $\nabla u \cdot n = g$
3. Then $\iiint_D f dV = \iint_{\partial D} g dS$

If this equality fails，the assumption that a solution u exists leads to a contradiction。

12. Check the validity of the maximum principle for the harmonic function
    $u(x,y) = \frac{1-x^2-y^2}{1-2x+x^2+y^2}$ in the disk $\overline{D} = \{x^2+y^2 \le 1\}$

u(x,y) is singular at (1,0)，where it becomes discontinuous。

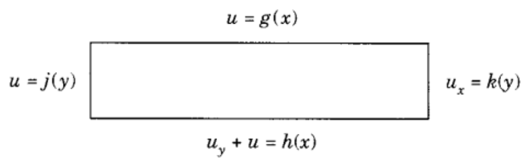
The maximum principle requires harmonicity in the open domain and continuity on the closure。 Since u fails to be continuous on the closed disk $\overline{D}$，the maximum principle does not apply。

13. A function u(x) is subharmonic in D if $\Delta u \ge 0$ in D。Prove that its maximum value is attained on bdyD。(Note that this is not true for the minimum value。)

§ 6.2 Rectangles and cubes

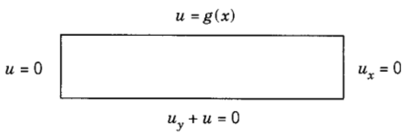
$\Delta u = u_{xx} + u_{yy} = 0$ in D。Where D is a rectangle {0<x<a,0<y<b}，on each sides one of of the standard boundary conditions is prescribed。(inhomogeneous Dirichlet，Neumann，or Robin)

Examples



1. Boundary conditions indicates as in the left figure。

2. For simplicity，assume h=0，j=0，k=0。



We separate the variables $u(x,y)=X(x)\cdot Y(y)$，then we get

$$\frac{X''}{X}+\frac{Y''}{Y}=0$$

Hence there is a constant λ such that $X''+\lambda X=0,$ for $0\le x\le a$，$Y''-\lambda Y=0$ for $0\le y\le b$

$X''=-\lambda X$ with $x(0)=X'(a)=0$

$Y''=\lambda Y$ with $Y'(0)+Y(0)=0$

3. …

Exerises

1. Solve $u_{xx}+u_{yy}=0$ in the rectangle 0<x<a，0<y<b with the following boundary conditions：

   $u_x=-a$ on x=0，$u_x=0$ on x=a

   $u_y=b$ on y=0，$u_y=0$ on y=b

   $$u(x,y)=\frac{1}{2}x^2-ax-\frac{1}{2}y^2+by+c$$

   DeepSeek 嘗試了很多方法，但 Walter A. Strauss 先生說，用猜的！
   U 是 x，y 的二次多項式，

2. Prove that the eigenfunctions $\{\sin my\sin nz\}$ are orthogonal on the square $\{0<y<\pi,0<z<\pi\}$

To prove the orthogonality of the eigenfunctions $\{\sin(my)\sin(nz)\}$on the square $\{0<y<\pi,0<z<\pi\}$ ，consider two distinct eigenfunctions $\sin(my)\sin(nz)$ and $\sin(py)\sin(qz)$ with(m,n)≠(p,q)。
The inner product is defined as:

$$\langle\sin(my)\sin(nz),\sin(py)\sin(qz)\rangle=\int_0^{\pi}\int_0^{\pi}\sin(my)\sin(nz)\sin(py)\sin(qz)\,dy\,dz.$$

This double integral separates into the product of two single-variable integrals：

$$\left(\int_0^{\pi} \sin(my)\sin(py)\,dy\right)\left(\int_0^{\pi} \sin(nz)\sin(qz)\,dz\right).$$

The orthogonality of sine functions on $[0,\pi]$ states that：

$$\int_0^{\pi} \sin(kx)\sin(lx)\,dx = \begin{cases} 0, & k \neq l, \\ \frac{\pi}{2}, & k = l. \end{cases}$$

Thus

1. If $m \neq p$ or $n \neq q$，at least one of the integrals vanishes，making the entire product zero。
2. If m=p and n=q，both integrals equal $\frac{\pi}{2}$, yielding $(\frac{\pi}{2})^2 \neq 0$

Hence，the eigenfunctions are orthogonal on the square。

Thus:

- If $m \neq p$ or $n \neq q$, at least one of the integrals vanishes, making the entire product zero.
- If $m = p$ and $n = q$, both integrals equal $\frac{\pi}{2}$, yielding $\left(\frac{\pi}{2}\right)^2 \neq 0$.

Hence, the eigenfunctions are orthogonal on the square.

3. Find the harmonic function u(x,y) in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with the boundary conditions：

   $u_y = 0$ for y=0 and for $y = \pi$

   u=0 for x=0 and $u = \cos^2 y = \frac{1}{2}(1+\cos 2y)$ for $x = \pi$

   Assume $u(x,y) = X(x)Y(y)$

   $$\nabla^2 u = 0 \Rightarrow \begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$$

   Boundary condition：$Y'(0) = Y'(\pi) = 0$

   $$Y'' + \lambda Y = 0 \Rightarrow Y = A\sin\sqrt{\lambda}y + B\cos\sqrt{\lambda}y$$

   $$Y'(0) = Y'(\pi) = 0 \Rightarrow \lambda_n = n^2, Y_n(y) = \cos(ny), n = 0,1,2,\ldots$$

   $X'' - n^2 X = 0 \Rightarrow X_n = Ae^{nx} + Be^{-nx}$，u(0,y)=0 $\Rightarrow A + B = 0$

   Hence $X_n = C_n \sinh(nx)$ (except n=0)

$$u(x,y)=\frac{A_0x}{2}+\sum_{n=1}^{\infty}C_n\sinh(nx)\cos(ny)$$

Apply boundary condition at $x=\pi$ ，

$$\frac{A_0\pi}{2}+\sum_{n=1}^{\infty}C_n\sinh(n\pi)\cos(ny)=\frac{1}{2}(1+\cos 2y)\Rightarrow A_0=\frac{1}{\pi},C_2=\frac{1}{2\sinh(2\pi)}$$

$$u(x,y)=\frac{x}{2\pi}+\frac{\sinh(2x)\cos(2y)}{2\sinh(2\pi)}$$

4. Find the harmonic function in the square {0<x<1，0<y<1}with the boundary

conditions $u(x,0)=x,u(x,1)=0,u_x(0,y)=0,u_x(1,y)=y^2$

Assume u(x,y)=v(x,y)+w(x,y)

v(x,0)=x,v(x,1)=0,v(0,y)=v(1,y)=0

$w_x(0,y)=0,w_x(1,y)=y^2$ ，w(x,0)=w(x,1)=0

分別解 v，w 然後相加。

其中用變數分離法解 v(x,y)，得 $v(x,y)=x-\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(n\pi x)\sinh(n\pi y)$

$w(x,y)=\sum_{n=1}^{\infty}C_n\cosh(n\pi x)\sinh(n\pi y)$ ，其中 $C_n=\dfrac{2(-1)^n}{\pi^3n^3\sinh(n\pi)}$

5.

5. Solve Example 1 in the case $b=1$, $g(x)=h(x)=k(x)=0$ but $j(x)$ an arbitrary function.

6. Solve the following Neumann problem in the cube $\{0<x<1, 0<y<1, 0<z<1\}$: $\Delta u=0$ with $u_z(x,y,1)=g(x,y)$ and homogeneous Neumann conditions on the other five faces, where $g(x,y)$ is an arbitrary function with zero average.

7. (a) Find the harmonic function in the semi-infinite strip $\{0\le x\le\pi, 0\le y<\infty\}$ that satisfies the "boundary conditions":

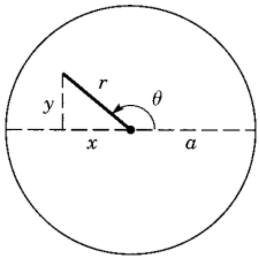
$$u(0,y)=u(\pi,y)=0,\quad u(x,0)=h(x),\ \lim_{y\to\infty}u(x,y)=0.$$

(b) What would go awry if we omitted the condition at infinity?

§ 6.3 Poisson formula

A much more interesting case is the Dirichlet problem for a circle。

The rotational invariance of Δ provides a hint that the circle is a natural shape for harmonic functions。



$$u_{xx}+u_{yy}=0 \quad \text{for} \quad x^2+y^2<a^2$$

$$u=h(\theta) \quad \text{for} \quad x^2+y^2=a^2$$

Saparate variables in polar coordinates：

$u=R(r)\Theta(\theta) \quad u_{xx}+u_{yy}=\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}$

$R''\Theta+\dfrac{1}{r}R'\Theta+\dfrac{1}{r^2}R\Theta''=0$ ， $\dfrac{r^2R''+rR'}{-R}=\dfrac{\Theta''}{\Theta}=-\lambda$

$r^2R''+rR'-\lambda R=0$ and $\Theta''+\lambda\Theta=0$

With BC： $\Theta(\theta+2\pi)=\Theta(\theta)$ for $-\infty<\theta<\infty$

Thus， $\lambda=n^2$ and $\Theta(\theta)=A\cos n\theta+B\sin n\theta$ n=1,2,3,⋯

There is also the solution $\lambda=0$ with $\Theta(\theta)=A$

⋯經過一番魔幻步驟，最後得到 Poisson formula：

$$u(r,\theta)=(a^2-r^2)\int_0^{2\pi}\frac{h(\phi)}{a^2-2ar\cos(\theta-\phi)+r^2}\frac{d\phi}{2\pi}.$$

另一形式是⋯

Exercises

1. Suppose that u is a harmonic function in the disk D={r<2} and that $u=3\sin 2\theta+1$ for r=2。Without finding the solution，answer the following questions

   (a) Find the maximum value of u in $\overline{D}$

   (b) Calculus the value of u at the origin。

2. Solve $u_{xx}+u_{yy}=0$ in the disk {r<a} with the boundary condition $u=1+3\sin\theta$ on r=a

   極座標的 Laplace equation： $\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0$

   用變數分離法 $u(r,\theta)=R(r)\Theta(\theta)$

   由於邊界條件是三角函數的形式，$\Theta(\theta)$ 滿足 $\dfrac{d^2\Theta}{d\theta^2}+\lambda\Theta=0$

   其一般解為 $\Theta_n(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta)$

   徑向方程為 $r^2R''+rR'-n^2R=0$

由 $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$ ， $R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0$

但 $\Theta''=-\lambda\Theta$ ，所以 $r^2R''+rR'-\lambda R=0$

$\because \Theta''+\lambda\Theta=0$ 其一般解為 $\Theta(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta)$ 是週期函數，λ 必須是平方數，即 $\lambda=n^2, n=0,1,2,\ldots$

徑向方程變成 $r^2R''+rR'-n^2R=0$

其一般解為 $R_n(r)=C_nr^n+D_nr^{-n}$ ，取 $D_n=0$ 以確保在 r=0 不發散。

邊界條件為 $u(a,\theta)=1+3\sin\theta$ ，展開傅立葉級數

$$1+3\sin\theta=A_0+\sum_{n=1}^{\infty}(A_n\cos n\theta+B_n\sin n\theta)\cdots$$

$$u(r,\theta)=1+\frac{3r}{a}\sin\theta$$

3. Solve $u_{xx}+u_{yy}=0$ in the disk {r<a} with the boundary condition $u=\sin^3\theta$

$$\sin^3\theta=\frac{3}{4}\sin\theta-\frac{1}{4}\sin 3\theta$$

$$u(r,\theta)=(\frac{3r}{4a})\sin\theta-(\frac{r^3}{4a^3})\sin 3\theta$$

4. Show that $P(r,\theta)$ is a harmonic function in D by using polar coordinates。That is，use $\Delta=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

$$P(r,\theta)=\frac{a^2-r^2}{a^2-2ar\cos\theta+r^2}=1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n\theta \qquad (17)$$

is the Poisson kernel. Note that P has the following three properties.

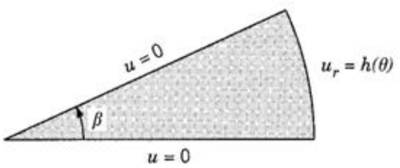
§ 6.4 circles，wedges，and annuli

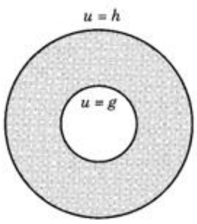
A wedge: $\{0<\theta<\theta_0, 0<r<a\}$

An annulus: $\{0<a<r<b\}$

The exterior of a circle: $\{a<r<\infty\}$

Examples





1. The wedge
2. The annlus
3. The exterior of a circle

Example 1 The wedge

Example The Annulus

Example

The exterior of a circle

Exercises

1. Solve $u_{xx} + u_{yy} = 0$ in the *exterior* $\{r > a\}$ of a disk, with the boundary condition $u = 1 + 3\sin\theta$ on $r = a$, and the condition at infinity that u be bounded as $r \to \infty$.

2. Solve $u_{xx} + u_{yy} = 0$ in the disk $r < a$ with the boundary condition
$$\frac{\partial u}{\partial r} - hu = f(\theta),$$
where $f(\theta)$ is an arbitrary function. Write the answer in terms of the Fourier coefficients of $f(\theta)$.

3. Determine the coefficients in the annulus problem of the text.

4. Derive Poisson's formula (9) for the exterior of a circle.

5. (a) Find the steady-state temperature distribution inside an annular plate $\{1 < r < 2\}$, whose outer edge $(r = 2)$ is insulated, and on whose inner edge $(r = 1)$ the temperature is maintained as $\sin^2\theta$. (Find explicitly all the coefficients, etc.)
   (b) Same, except $u = 0$ on the outer edge.

6. Find the harmonic function u in the semidisk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on the diameter $(\theta = 0, \pi)$ and
$$u = \pi\sin\theta - \sin 2\theta \quad \text{on } r = 1.$$

7. Solve the problem $u_{xx} + u_{yy} = 0$ in D, with $u = 0$ on the two straight sides, and $u = h(\theta)$ on the arc, where D is the wedge of Figure 1, that is, a sector of angle β cut out of a disk of radius a. Write the solution as a series, but don't attempt to sum it.

8. An annular plate with inner radius a and outer radius b is held at temperature B at its outer boundary and satisfies the boundary condition $\partial u/\partial r = A$ at its inner boundary, where A and B are constants. Find the temperature if it is at a steady state. (*Hint:* It satisfies the two-dimensional Laplace equation and depends only on r.)

9. Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a, 0 < \theta < \beta$ with the BCs

$$u = \theta \quad \text{on } r = a, \qquad u = 0 \quad \text{on } \theta = 0, \qquad \text{and} \qquad u = \beta \quad \text{on } \theta = \beta.$$

(*Hint:* Look for a function independent of r.)

10. Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the following BCs:

$$u = 0 \quad \text{on } x = 0 \text{ and on } y = 0 \quad \text{and} \quad \frac{\partial u}{\partial r} = 1 \quad \text{on } r = a.$$

Write the answer as an infinite series and write the first two nonzero terms explicitly.

11. Prove the uniqueness of the Robin problem

$$\Delta u = f \quad \text{in } D, \qquad \frac{\partial u}{\partial n} + au = h \quad \text{on bdy } D,$$

where D is any domain in three dimensions and where a is a positive constant.

12. (a) Prove the following still stronger form of the maximum principle, called the Hopf form of the maximum principle. If $u(\mathbf{x})$ is a nonconstant harmonic function in a connected plane domain D with a smooth boundary that has a maximum at $\mathbf{x}_0$ (necessarily on the boundary by the strong maximum principle), then $\partial u/\partial n > 0$ at $\mathbf{x}_0$ where $\mathbf{n}$ is the unit *outward* normal vector. (This is difficult: see [PW] or [Ev].)
(b) Use part (a) to deduce the uniqueness of the Neumann problem in a connected domain, up to constants.

13. Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta, a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.