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Ch7 Fourier Transforms

7.1 The Fourier Transform

7.1.1. Find the Fourier transform of the following functions:

(a) $e^{-(x+4)^2}$, (b) $e^{-|x+1|}$, (c) $\begin{cases} x, & |x|<1, \\ 0, & \text{otherwise}, \end{cases}$ (d) $\begin{cases} e^{-2x}, & x \geq 0, \\ e^{3x}, & x \leq 0, \end{cases}$

(e) $\begin{cases} e^{-|x|}, & |x| \geq 1, \\ e^{-1}, & |x| \leq 1, \end{cases}$ (f) $\begin{cases} e^{-x}\sin x, & x>0, \\ 0, & x \leq 0, \end{cases}$ (g) $\begin{cases} 1-|x|, & |x| \leq 1, \\ 0, & \text{otherwise}. \end{cases}$

7.1.2. Find the Inverse Fourier transform of the following functions: (a) e^{-k^2}, (b) $e^{-|k|}$,

(c) $\begin{cases} e^{-k}\sin k, & k \geq 0, \\ 0, & k \leq 0, \end{cases}$ (d) $\begin{cases} 1, & \alpha<k<\beta, \\ 0, & \text{otherwise}, \end{cases}$ (e) $\begin{cases} 1-|k|, & |k|<1, \\ 0, & \text{otherwise}. \end{cases}$

7.1.3. Find the inverse Fourier transform of the function $1/(k+c)$ when (a) $c=a$ is real; (b) $c=\mathrm{i}\,b$ is purely imaginary; (c) $c=a+\mathrm{i}\,b$ is an arbitrary complex number.

7.1.4. Find the inverse Fourier transform of $1/(k^2-a^2)$, where $a>0$ is real. *Hint*: Use Exercise 7.1.3.

7.1.5. (a) Find the Fourier transform of $e^{\mathrm{i}\,\omega x}$. (b) Use this to find the Fourier transforms of the basic trigonometric functions $\cos\omega x$ and $\sin\omega x$.

7.1.6. Write down two real integral identites that result from the inverse Fourier transform of (7.28).

7.1.7. Write down two real integral identities that follow from (7.17).

7.1.8. (a) Find the Fourier transform of the hat function $f_n(x) = \begin{cases} n-n^2|x|, & |x| \leq 1/n, \\ 0, & \text{otherwise}. \end{cases}$
(b) What is the limit, as $n\to\infty$, of $\widehat{f}_n(k)$?
(c) In what sense is the limit the Fourier transform of the limit of $f_n(x)$?

7.1.9. (a) Justify the linearity of the Fourier transform, as in (7.11).
(b) State and justify the linearity of the inverse Fourier transform.

7.1.10. If the Fourier transform of $f(x)$ is $\widehat{f}(k)$, prove that (a) the Fourier transform of $f(-x)$ is $\widehat{f}(-k)$; (b) the Fourier transform of the complex conjugate function $\overline{f(x)}$ is $\overline{\widehat{f}(-k)}$.

7.1.11. *True or false*: If the complex-valued function $f(x) = g(x) + \mathrm{i}\,h(x)$ has Fourier transform $\widehat{f}(k) = \widehat{g}(k) + \mathrm{i}\,\widehat{h}(k)$, then $g(x)$ has Fourier transform $\widehat{g}(k)$ and $h(x)$ has Fourier transform $\widehat{h}(k)$.

7.1.12. (*a*) Prove that the Fourier transform of an even function is even. (*b*) Prove that the Fourier transform of a real even function is real and even. (*c*) What can you say about the Fourier transform of an odd function? (*d*) Of a real odd function? (*e*) What about a general real function?

7.1.13. Prove the Shift Theorem 7.4.

7.1.14. Prove the Dilation Theorem 7.5.

7.1.15. Given that the Fourier transform of $f(x)$ is $\widehat{f}(k)$, find, from first principles, the Fourier transform of $g(x) = f(a\,x + b)$, where a and b are fixed real constants.

7.1.16. Let a be a real constant. Given the Fourier transform $\widehat{f}(k)$ of $f(x)$, find the Fourier transforms of (a) $f(x)\,e^{\mathrm{i}\,a\,x}$, (b) $f(x)\cos a\,x$, (c) $f(x)\sin a\,x$.

7.1.17. A common alternative convention for the Fourier transform is to define

$$\widehat{f}_1(k) = \int_{-\infty}^{\infty} f(x)\, e^{-\,\mathrm{i}\,k\,x}\, dx.$$

(*a*) What is the formula for the corresponding inverse Fourier transform?
(*b*) How is $\widehat{f}_1(k)$ related to our Fourier transform $\widehat{f}(k)$?

7.1.18. Another convention for the Fourier transform is to define $\widehat{f}_2(k) = \int_{-\infty}^{\infty} f(x)\, e^{-\,2\pi\,\mathrm{i}\,k\,x}\, dx$. Answer the questions in Exercise 7.1.17 for this version of the Fourier transform.

7.1.19. The *cosine* and *sine transforms* of a real function $f(x)$ are defined as

$$\widehat{c}\,(k) = \int_{-\infty}^{\infty} f(x) \cos k\,x\, dx, \qquad \widehat{s}\,(k) = \int_{-\infty}^{\infty} f(x) \sin k\,x\, dx. \tag{7.40}$$

(*i*) Prove that $\widehat{f}(k) = \widehat{c}\,(k) - \mathrm{i}\,\widehat{s}\,(k)$. (*ii*) Find the cosine and sine transforms of the functions in Exercise 7.1.1. (*iii*) Show that $\widehat{c}\,(k)$ is an even function, while $\widehat{s}\,(k)$ is an odd function. (*iv*) Show that if f is an even function, then $\widehat{s}\,(k) \equiv 0$, while if f is an odd function, then $\widehat{c}\,(k) \equiv 0$.

7.1.20. The *two-dimensional Fourier transform* of a function $f(x,y)$ defined for $(x,y) \in \mathbb{R}^2$ is

$$\widehat{f}(k,l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, e^{-\,\mathrm{i}\,(k\,x + l\,y)}\, dx\, dy. \tag{7.41}$$

(*a*) Compute the Fourier transform of the following functions:
(*i*) $e^{-\,|\,x\,| - |\,y\,|}$; (*ii*) $e^{-x^2 - y^2}$; (*iii*) the delta function $\delta(x-\xi)\,\delta(y-\eta)$,
(*iv*) $\begin{cases} 1, & |\,x\,|, |\,y\,| \le 1, \\ 0, & \text{otherwise}, \end{cases}$ (*v*) $\begin{cases} 1, & |\,x\,| + |\,y\,| \le 1, \\ 0, & \text{otherwise}, \end{cases}$ (*vi*) $\cos(x-y)$.
(*b*) Show that if $f(x,y) = g(x)\,h(y)$, then $\widehat{f}(k,l) = \widehat{g}(k)\,\widehat{h}(l)$.
(*c*) What is the formula for the inverse two-dimensional Fourier transform, i.e., how can you reconstruct $f(x,y)$ from $\widehat{f}(k,l)$?

7.2 Derivatives and Integrations

7.2.1. Determine the Fourier transform of the following functions:
(a) $e^{-x^2/2}$, (b) $x\,e^{-x^2/2}$, (c) $x^2\,e^{-x^2/2}$, (d) x, (e) $x\,e^{-2\,|\,x\,|}$, (f) $x\tan^{-1}x$.

7.2.2. Find the Fourier transform of (a) the error function $\operatorname{erf} x = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-z^2}\,dz$;
(b) the complementary error function $\operatorname{erfc} x = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_x^\infty e^{-z^2}\,dz$.

7.2.3. Find the inverse Fourier transform of the following functions:
(a) k, (b) $k\,e^{-k^2}$, (c) $\dfrac{k}{(1+k^2)^2}$, (d) $\dfrac{k^2}{k-\mathrm{i}}$, (e) $\dfrac{1}{k^2-k}$.

7.2.4. Is the usual formula $\sigma'(x)=\delta(x)$ relating the step and delta functions compatible with their Fourier transforms? Justify your answer.

7.2.5. Find the Fourier transform of the derivative $\delta'(x)$ of the delta function in three ways: (a) First, directly from the definition of $\delta'(x)$; (b) second, using the formula for the Fourier transform of the derivative of a function; (c) third, as a limit of the Fourier transforms of the derivatives of the functions in Exercise 7.1.8. (d) Are your answers all the same? If not, can you explain any discrepancies?

7.2.6. Show that one can obtain the Fourier transform of the Gaussian function $f(x)=e^{-x^2/2}$ by the following trick. First, prove that $\widehat{f}\,'(k) = -k\,\widehat{f}(k)$. Use this to deduce that $\widehat{f}(k) = c\,e^{-k^2/2}$ for some constant c. Finally, use the Symmetry Principle to determine c.

7.2.7. If $f(x)$ has Fourier transform $\widehat{f}(k)$, which function has Fourier transform $\dfrac{\widehat{f}(k)}{k}$?

7.2.8. If $f(x)$ has Fourier transform $\widehat{f}(k)$, what is the Fourier transform of $\dfrac{f(x)}{x}$?

7.2.9. Use Exercise 7.2.8 to find the Fourier transform of
(a) $1/x$, (b) $x^{-1}e^{-|\,x\,|}$, (c) $x^{-1}e^{-x^2}$, (d) $(x^3+4x)^{-1}$.

7.2.10. Directly justify formula (7.43) by integrating the relevant Fourier transform integral by parts. What do you need to assume about the behavior of $f(x)$ for large $|\,x\,|$?

7.2.11. Given the Fourier transform $\widehat{f}(k)$ of $f(x)$, find the Fourier transform of its integral $g(x)=\displaystyle\int_a^x f(y)\,dy$ starting at the point $a\in\mathbb{R}$.

7.2.12. (a) Explain why the Fourier transform of a 2π–periodic function $f(x)$ is a linear combination of delta functions, $\widehat{f}(k) = \sum_{n=-\infty}^{\infty} c_n\,\delta(k-n)$, where c_n are the (complex) Fourier series coefficients (3.65) of $f(x)$ on $[-\pi,\pi]$.
(b) Find the Fourier transform of the following periodic functions:
(i) $\sin 2x$, (ii) $\cos^3 x$, (iii) the 2π–periodic extension of $f(x) = x$,
(iv) the sawtooth function $h(x) = x \bmod 1$, i.e., the fractional part of x.

7.2.13. Determine the Fourier transforms of (a) $\cos x - 1$, (b) $\dfrac{\cos x - 1}{x}$, (c) $\dfrac{\cos x - 1}{x^2}$.
Hint: Use Exercises 7.2.8 and 7.2.12.

7.2.14. Write down the formulas for differentiation and integration for the alternative Fourier transforms of Exercises 7.1.17 and 7.1.18.

7.2.15. (a) What is the two-dimensional Fourier transform, (7.41), of the gradient $\nabla f(x,y)$ of a function of two variables?
(b) Use your formula to find the Fourier transform of the gradient of $f(x,y) = e^{-x^2-y^2}$.

7.3 Green's Functions and Convolution

7.3.1. Use partial fractions to compute the inverse Fourier transform of the following rational functions. *Hint*: First solve Exercise 7.1.3.

(a) $\dfrac{1}{k^2 - 5k - 6}$, (b) $\dfrac{e^{\,i\,k}}{k^2 - 1}$, (c) $\dfrac{1}{k^4 - 1}$, (d) $\dfrac{\sin 2k}{k^2 + 2k - 3}$.

7.3.2. Find the inverse Fourier transform of the function $\dfrac{1}{k^2 + 2k + 5}$:
(a) using partial fractions; (b) by completing the square. Are your answers the same?

7.3.3. Use partial fractions to compute the Fourier transform of the following functions:
(a) $\dfrac{1}{x^2 - x - 2}$, (b) $\dfrac{1}{x^3 + x}$, (c) $\dfrac{\cos x}{x^2 - 9}$.

7.3.4. Find a solution to the differential equation $-\dfrac{d^2u}{dx^2} + 4u = \delta(x)$ using the Fourier transform.

7.3.5. Use the Fourier transform to solve the boundary value problem
$-u'' + u = \delta'(x-1)$ for $-\infty < x < \infty$, with $u(x) \to 0$ as $x \to \pm\infty$.

7.3.6. (a) Use the Fourier transform to solve (7.48) with $h(x) = e^{-|x|}$ when $\omega = 1$.
(b) Verify that your solution can be obtained as a limit of (7.51) as $\omega \to 1$.

7.3.7. Use the Fourier transform to find a bounded solution to the differential equation

$$u'''' + u = e^{-2\,|\,x\,|}.$$

7.3.8. Use the Fourier transform to find an integral formula for a bounded solution to the Airy differential equation $-\dfrac{d^2u}{dx^2} = x\,u$.

7.3.9. Prove that (7.51) is a twice continuously differentiable function of x and satisfies the differential equation (7.48).

7.3.10. (a) Find the Fourier transform of the convolution $h(x) = f_e * g(x)$ of an even exponential pulse $f_e(x) = e^{-\,|\,x\,|}$ and a Gaussian $g(x) = e^{-x^2}$. (b) What is $h(x)$?

7.3.11. What is the convolution of a Gaussian kernel e^{-x^2} with itself? *Hint*: Use the Fourier transform.

7.3.12. Find the function whose Fourier transform is $\widehat{f}(k) = (k^2+1)^{-2}$.

7.3.13. (a) Write down the Fourier transform of the box function $f(x) = \begin{cases} 1, & |\,x\,| < \frac{1}{2}, \\ 0, & |\,x\,| > \frac{1}{2}. \end{cases}$
(b) Graph the hat function $h(x) = f * f(x)$ and find its Fourier transform.
(c) Determine the *cubic B spline* $s(x) = h * h(x)$ and its Fourier transform.

7.3.14. Let $f(x) = \begin{cases} \sin x, & 0 < x < \pi, \\ 0, & \text{otherwise}, \end{cases}$ $\quad g(x) = \begin{cases} \cos x, & 0 < x < \pi, \\ 0, & \text{otherwise}. \end{cases}$
(a) Find the Fourier transforms of $f(x)$ and $g(x)$; (b) compute the convolution $h(x) = f * g(x)$; (c) find its Fourier transform $\widehat{h}(k)$.

7.3.15. Use convolution to find an integral formula for the function whose Fourier transform is

$$\text{(a)}\ \frac{e^{-k^2}}{k^2+1}, \qquad \text{(b)}\ \frac{\sin k}{k(k^2+1)}, \qquad \text{(c)}\ \frac{\sin^2 k}{k^2}, \qquad \text{(d)}\ \frac{\operatorname{sign} k}{1+\mathrm{i}\,k}.$$

If possible, evaluate the resulting convolution integral.

7.3.16. Let $f(x)$ be a smooth function. (a) Find its convolution $\delta' * f$ with the derivative of the delta fiunction. (b) More generally, find $\delta^{(n)} * f$.

7.3.17. According to Proposition 7.7, the Fourier transform of the derivative $f'(x)$ is obtained by multiplying $\widehat{f}(k)$ by $\mathrm{i}\,k$. Can you reconcile this result with the Convolution Theorem 7.13?

7.3.18. The *Hilbert transform* of a function $f(x)$ is defined as the integral

$$h(x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(\xi)\,d\xi}{\xi - x}. \tag{7.57}$$

Find a formula for its Fourier transform $\widehat{h}(k)$ in terms of $\widehat{f}(k)$. *Remark*: The bar on the integral indicates the *principal value integral*, [**2**], which is $\lim_{\delta \to 0^+} \left(\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty} \right) \frac{f(\xi)\,d\xi}{\xi - x}$, and is employed to avoid the integral diverging at the singular point $x = \xi$.

7.3.19. Use the Fourier transform to solve the integral equation $\int_{-\infty}^{\infty} e^{-|x-\xi|}\,u(\xi)\,d\xi = f(x)$. Then verify your solution when $f(x) = e^{-2|x|}$.

7.3.20. Suppose that $f(x)$ and $g(x)$ are identically 0 for all $x < 0$. Prove that their convolution product $h = f * g$ reduces to a finite integral: $h(x) = \begin{cases} \int_0^x f(x-\xi)\,g(\xi)\,d\xi, & x > 0, \\ 0, & x \le 0. \end{cases}$

7.3.21. Given that the support of $f(x)$ is contained in the interval $[a, b]$ and the support of $g(x)$ is contained in $[c, d]$, what can you say about the support of their convolution $h(x) = f * g(x)$?

7.3.22. Prove the convolution properties (a–e).

7.3.23. In this exercise, we explain how convolution can be used to smooth out rough data. Let $g_\varepsilon(x) = \dfrac{\varepsilon}{\pi(\varepsilon^2 + x^2)}$. (a) If $f(x)$ is any (reasonable) function, show that $f_\varepsilon(x) = g_\varepsilon * f(x)$ for $\varepsilon \neq 0$ is a C^∞ function. (b) Show that $\lim_{\varepsilon \to 0} f_\varepsilon(x) = f(x)$.

7.3.24. Explain why the Shift Theorem 7.4 is a special case of the Convolution Theorem 7.13.

7.3.25. Suppose $f(x)$ and $g(x)$ are 2π–periodic and have respective complex Fourier coefficients c_k and d_k. Prove that the complex Fourier coefficients e_k of the product function $f(x)\,g(x)$ are given by the *convolution summation* $e_k = \sum_{j=-\infty}^{\infty} c_j\,d_{k-j}$. *Hint*: Substitute the formulas for the complex Fourier coefficients into the summation, making sure to use two different integration variables, and then use (6.37).

7.4 The Fourier Transform on Hilbert Space

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