Lctures on PDE   Lecture 8 harmonic functions   V. I. Arnold

1. The function $u(x,y,z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$ is harmonic everywhere except at the

origin。(與位勢理論有關)

$$\frac{\partial u}{\partial x} = -x(x^2 + y^2 + z^2)^{-3/2} \quad , \quad \frac{\partial^2 u}{\partial x^2} = -(x^2 + y^2 + z^2)^{-3/2} + 3x^2(x^2 + y^2 + z^2)^{-5/2}$$

靜電場 電位 $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r}$ ，重力場 $\phi = -\dfrac{GM}{r}$

定理 1
Every harmonic function in a simply connected domain of the plane is the real part of a
holomorphic function (全純函數)which is defined in that domain up to an additive purely
imaginary constant。
在單連通區域D 上，任何調和函數 u 都是某個全純函數 f 的實部，且f 在相差一
個純虛常數的意義下唯一。
找一個 holomorphic function f=u+iv，其中 u 是給定的 harmonic function，要找 v。
考慮 $F = (-\dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial x})$ for a 2-dim vector field F=(P,Q) $\ curF = \dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$

So $curF = \dfrac{\partial}{\partial x}(\dfrac{\partial u}{\partial x}) - (-\dfrac{\partial}{\partial y}(\dfrac{\partial u}{\partial y})) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$
在單連通區域中 cur=0 的向量場 $\Rightarrow$ 保守場
(在 R^3 中，對於一個向量場 F，封閉對應的就是 $\nabla \times F = 0$(無旋場))

存在 v 使得 $\nabla v = F = (-u_y, u_x)$

Let f=u+iv then f holomorphic and Re(f)=u

若 v_1, v_2 皆為 harmonic conjugate 則 $\nabla(v_1 - v_2) = 0 \quad v_1 - v_2 = const$

Arnold 是用 differential form 的方式解說：

考慮 $\alpha = (-u_y)dx + (u_x)dy$ 則 $d\alpha = -u_{yy}dy \wedge dx + u_{xx}dx \wedge dy = (u_{xx} + u_{yy})dx \wedge dy = 0$

又 $u_x = v_y, u_y = -v_x$ ， $\alpha = (-u_y)dx + (u_x)dy = v_x dx + v_y dy = dv$

$d\alpha = 0$ 區域的單連通性保證 v 的存在。
在一個單連通區域，一個 closed form 是 exact form。

§ A holomorphic function f(z)：

(1) 複可微

(2) 滿足 Cauchy-Riemann 方程 f(z)=f(x+iy)=u(x,y)+iv(x,u) $u_x = v_y, u_y = -v_x$

例如

(1) $f(z) = z^2$

(2) $f(z) = e^z$

(3) $f(z) = \dfrac{az+b}{cz+d}, a,b,c,d \in C, ad - bc \neq 0$

(4) f(z)=log(z)是多值函數，不是在整個 $C/\{0\}$ 上 holomorphi

   $z = re^{i\theta}$ $\log z = \log r + i(\theta + 2k\pi)$

   …

Log(z)不是定義在平面上 而是定義在 Riemann surface 上的單值 holomorphic 函數。

在單連通區域 $U \subset C - \{0\}$ 上，一定可以定義一個 holomorphic 的 log(z)：

在單連通區域上，若 f 是 holomorphic 則對所有閉曲線 γ ，$\oint_\gamma f(z)dz = 0 \Leftrightarrow$ f 有原始函數。

$\dfrac{1}{z}$ 在 $C - \{0\}$ 是 holomorphic，但是它在整個 $C/\{0\}$ 上沒有原始函數，因為

$$\oint_{|z|=1} \frac{1}{z} dz = 2\pi i$$

定義 $\log z := \int_{z_0}^z \dfrac{1}{\zeta} d\zeta$ 因為與路徑無關 所以是 well-defined 且 $(\log z)' = \dfrac{1}{z}$

$\overline{\delta} = \dfrac{\partial}{\partial \overline{z}} = \dfrac{1}{2}(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y})$ 稱為 D-bar operator(or CR operator)

f 是全純函數 $\Leftrightarrow \overline{\delta}f = 0$。$\overline{\delta}$ 是二維 PDE 的重要工具。

回到原定理：找 f=u+iv 全純，即 $\dfrac{\partial f}{\partial \overline{z}} = 0$ (u 已知，v 未知)

$\dfrac{\partial f}{\partial \overline{z}} = \dfrac{\partial(u+iv)}{\partial \overline{z}} = \dfrac{\partial u}{\partial \overline{z}} + i\dfrac{\partial v}{\partial \overline{z}} = 0$

$\dfrac{\partial v}{\partial \overline{z}} = i\dfrac{\partial u}{\partial \overline{z}}$ (因為 u 是 real value 所以 $\dfrac{\partial u}{\partial \overline{z}} = \overline{(\dfrac{\partial u}{\partial z})}$)

Let $\varphi(z) := \dfrac{\partial u}{\partial \overline{z}}$ 原命題就是要找 $\dfrac{\partial v}{\partial \overline{z}} = i\varphi(z)$ 的解 v(z) 這是一個 $\overline{\delta}$ 非齊次方程

$$\frac{\partial}{\partial \bar{z}}(\bar{z}\varphi(z)) = \varphi(z) + \bar{z}\frac{\partial \varphi(z)}{\partial \bar{z}} = \varphi(z) \quad (因為 \Delta u = 0 \Leftrightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0)$$

$$\frac{\partial}{\partial \bar{z}}(i\bar{z}\varphi(z)) = i\varphi(z)$$

$$v_0(z) = i\bar{z}\varphi(z) = i\bar{z}\frac{\partial u}{\partial \bar{z}} \ 即為 \ f \ 所需的 \ v \ 的一個特別解。$$

或者 由 $\frac{\partial v}{\partial \bar{z}} = i\frac{\partial u}{\partial \bar{z}}$ 因為 simply connected

$$v(z) := \text{Im}(\int_{z_0}^{z} 2i\frac{\partial u}{\partial \zeta}d\zeta) \ 即可。$$

§ 6.1 Laplace equation $\quad \Delta u = u_{xx} + u_{yy} = 0 \quad\quad$ Walter A. Strauss

或者寫成 $\Delta u = \nabla \cdot \nabla u$ (subharmonic $\Leftrightarrow \Delta u \geq 0$)

$u : \Omega \subset R^n \to R$ (or C) 滿足 $\Delta u=0$ 則 u 稱為 harmoic function。

$E(u) = \int_{\Omega}|\nabla u|^2 dx$ 是能量泛函。

$\Delta u = f$ with a given function is called Poisson equation。

$$f : (M, g) \to (N, h)$$

$$\forall p \in M, \exists \lambda(p) > 0 \ni f^*h = \lambda(p)^2 g \ \text{ i.e.}$$

$$\langle df_p(v), df_p(w) \rangle_h = \lambda(p)^2 <v,w>_g \ \ 則稱 \ f \ 為 \ \text{conformal map}$$

若 $f : M \to R^3$ 是共形且調和，則其像為 minimal surface。

例如懸鏈面(catenoid)

f(u,v)=(coshucosv,coshusinv,u)

$$I = \cosh^2 u(du^2 + dv^2) \ \cdots 表示 \ \text{conformal}$$

Δ (the **Laplace operator**) is defined as： $\Delta u = \sum_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2}$

Polar coordinates (r, θ)：

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Spherical coordinates (r, θ, ϕ)： $\Delta u = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2\dfrac{\partial u}{\partial r}) + ...$

Laplace-Beltrami operator Δ_g is a generalization of the Laplace operator to functions defined on Riemannian manifolds。

$$f : M \to R \text{，} \Delta_g f = div_g(grad_g f) = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f)$$

in local coordinates $(x^1, ..., x^n)$

$f : \Omega \subset R^n \to R$

Key properties：
1. Mean value property

    For any ball $B(x,r) \subset \Omega$， $u(x) = \dfrac{1}{|\partial B(x,r)|}\int_{\partial B} u(y)dS(y)$

2. Maximum principle

3. Smoothness
4. Liouville theorem

    A bounded harmonic function on R^n must be constant。

    There is no non-constant negative harmonic function defined on the Euclidean space。

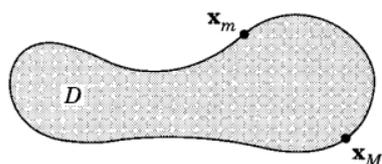   There is no non-constant negative subharmonic function on R^2。

Examples：
2. Electrostatics(靜電學)

    curlE=0， $divE = 4\pi\rho$

For the electric potential ϕ，$\Delta\phi = div(grad\phi) = -div\mathbf{E} = -4\pi\rho$

§ maximum principle

Let D be a conncted bounded open set。Let u(x,y,z) be a harmonic function in D that is continuous on $\overline{D}$ $(= D \cup \partial D)$。Then the maximum and the minimum values of u are attained on ∂D and nowhere inside。(unless $u \equiv$ conatsnt)

有朋自遠方來 訪問 Robert Finn 提到 Eberhard Hopf 的 strong maximum principle。

恰好看到此章提到 maximum principle。

[maximum principle $u_t = ku_{xx}$ is a one-dimensional **diffusion equation** PDE102-

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§ rotational invariance

The Laplace equation is invariant under all rigid motions。

In engineering the Laplacian is a model for isotropic physical situations，in which there is no preferred direction。

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial f}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} \quad , \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \quad \cdots(*)$$

$x = r\cos\theta, y = r\sin\theta$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial}{\partial\theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$r = \sqrt{x^2 + y^2}, \theta = \tan^{-1}\frac{y}{x}$ to find $\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial\theta}{\partial x}, \frac{\partial\theta}{\partial y}$

例如 $\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta$ ， $\frac{\partial\theta}{\partial x} = \frac{-y/x^2}{1+(y/x)^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} = \frac{-\sin\theta}{r}$

$$\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \quad , \quad \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$u_x = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x} = u_r\cos\theta - u_\theta\frac{\sin\theta}{r}$$

$$u_y = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y} = u_r\sin\theta + u_\theta\frac{\cos\theta}{r}$$

$$u_{xx} = \frac{\partial}{\partial x}(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}) = (\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta})(u_r \cos\theta - u_\theta \frac{\sin\theta}{r})$$

$$= \cos\theta \frac{\partial}{\partial r}(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}) - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}(u_r \cos\theta - u_\theta \frac{\sin\theta}{r})$$

$$= \cos^2\theta\, u_{rr} + \cos\theta(-\frac{\sin\theta}{r}u_{\theta r} + \frac{\sin\theta}{r^2}u_\theta) + \frac{\sin^2\theta}{r}u_r - \frac{\sin\theta\cos\theta}{r}u_{r\theta} + \frac{\sin\theta\cos\theta}{r^2}u_\theta + \frac{\sin^2\theta}{r^2}u_{\theta\theta}$$

$$= \cos^2\theta\, u_{rr} - \frac{2\sin\theta\cos\theta}{r}u_{r\theta} + \frac{\sin^2\theta}{r}u_r + \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\sin^2\theta}{r^2}u_{\theta\theta}$$

同理 $u_{yy} = (\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta})(u_r \sin\theta + u_\theta \frac{\cos\theta}{r})$

$=\cdots$

$$= \sin^2\theta\, u_{rr} - \frac{2\sin\theta\cos\theta}{r^2}u_\theta + \frac{\cos^2\theta}{r}u_r + \frac{2\sin\theta\cos\theta}{r}u_{r\theta} + \frac{\cos^2\theta}{r^2}u_{\theta\theta}$$

合併得

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

若 harmonic functions 本身是旋轉不變，則(*)變成 $0 = u_{rr} + \frac{1}{r}u_r$，若 u 與 θ 無關，

此方程變成 $(ru_r)_r = 0$，$u = c_1 \ln r + c_2$

後面證明 3 維 Laplacian 在空間剛體運動下皆為不變量(暫略)。

習作

1. Show that a function which is a power series in the complex variable x+iy must satisfy the Cauchy–Riemann equations and therefore Laplace equation。

    z=x+iy，f(z)=u(z)+iv(z)=u(x,y)+iv(x,y) ，$f(z) = \sum_{n=0}^{\infty} a_n z^n$

    因為冪級數在收斂區域內逐項可微，因此對 x 和 y 微分：

    $$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \sum_{n=1}^{\infty} a_n n(x+iy)^{n-1}, \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = i\sum_{n=1}^{\infty} a_n n(x+iy)^{n-1}$$

    $i(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}) = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}$，可以得到 Cauchy-Riamann equations：

    $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$。

    $u_{xx} = v_{yx} = v_{xy} = -u_{yy}$ then $\Delta u = 0$

    同理 $\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -(\frac{\partial^2 u}{\partial y \partial x}) + \frac{\partial^2 u}{\partial x \partial y} = 0$

2. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} + u_{zz} = k^2 u$，where

   k is a positive constant。

   球坐標系 (r, θ, ϕ) 下，$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta$

   Laplacian 的表式為：

   $$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 f}{\partial \phi^2}$$

   與 θ 無關時，Laplacian 用球面座標表示為 $\nabla^2 u = \frac{1}{r^2} \frac{d}{dr}(r^2 \frac{du}{dr})$

   $\frac{1}{r^2} \frac{d}{dr}(r^2 \frac{du}{dr}) = k^2 u \Rightarrow \frac{d}{dr}(r^2 \frac{du}{dr}) = k^2 r^2 u$，let v(r)=ru(r)，$u = \frac{v}{r} \Rightarrow \frac{du}{dr} = \frac{v'}{r} - \frac{v}{r^2}$

   $r^2 \frac{du}{dr} = rv' - v$，$\frac{d}{dr}(r^2 \frac{du}{dr}) = \frac{d}{dr}(rv' - v) = v' + rv'' - v' = rv''$

   $rv'' = k^2 r^2 u = k^2 r^2 \times \frac{v}{r} = k^2 rv \Rightarrow v'' = k^2 v$

   $v(r) = Ae^{kr} + Be^{-kr} \quad u = \frac{Ae^{kr} + Be^{-kr}}{r}$

3. Find the solutions that depend only on r of the equation $u_{xx} + u_{yy} = k^2 u$，where k is a

   positive constant。

   The given equation is the Helmholtz equation。

   $\frac{1}{r} \frac{\partial}{\partial r}(r \frac{\partial u}{\partial r}) = k^2 u \Rightarrow u_{rr} + \frac{1}{r} u_r = k^2 u \cdots(1)$

   Let s=kr，(1)兩邊同乘以 r^2，the equation becomes：

   $s^2 \frac{d^2 u}{ds^2} + s \frac{du}{ds} - s^2 u = 0$ （a modified Bessel differential equation）

   $u(r) = c_1 I_0(kr) + c_2 K_0(kr)$

where I_0 and K_0 are the modified Bessel functions of the first and second kind，

> The Bessel differential equation：
>
> $$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - v^2)y = 0$$
>
> General solution is $y(x) = c_1 J_v(x) + c_2 Y_v(x)$
>
> The modified Bessel differential equation：

respectively，and c_1, c_2 are constants。

4. Solve $u_{xx} + u_{yy} + u_{zz} = 0$ in the spherical shell 0<a<r<b with the boundary condition

u=A on r=a and u=B on r=b，where A and B are constants。

$$\frac{1}{r^2} \frac{d}{dr}(r^2 \frac{du}{dr}) = 0 \Rightarrow u = -\frac{c_1}{r} + c_2 \Rightarrow u = \frac{C_1}{r} + c_2$$

At r=A，$A = \frac{C_1}{a} + c_2$ ；at r=B，$B = \frac{C_1}{b} + c_2$ 解出C_1, c_2

$$u(r) = \frac{Aa(b-r) + Bb(r-a)}{r(b-a)}$$

5. Solve $u_{xx} + u_{yy} = 1$ in r<a with u(x,y) vanishing on r=a。

$$\frac{1}{r} \frac{d}{dr}(r \frac{du}{dr}) = 1 \text{ with u(a)=0}$$

$$u(r) = \frac{1}{4}(r^2 - a^2)$$

6. Solve $u_{xx} + u_{yy} = 1$ in the annulus(圓環) a<r<b with u(x,y) vanishing on both parts of

the boundary r=a an r=b。

$$\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr}) = 1 \Rightarrow u(r) = \frac{1}{4}r^2 + c\ln r + d \quad \text{with u(a)=u(b)=0}$$

$$u(r) = \frac{1}{4}\{r^2 - \frac{b^2\ln(\frac{r}{a}) + a^2\ln(\frac{b}{r})}{\ln(\frac{b}{a})}\}$$

7. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell a<r<b with u(x,y,z) vanishng on both the inner and outer boundaries。

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = 1$$

$$u(r) = \frac{1}{6}r^2 - \frac{c_1}{r} + c_2 \text{，set u(a)=u(b)=0 解 } c_1 = \frac{-ab(a+b)}{6}, c_2 = -\frac{a^2+ab+b^2}{6}$$

$$u(r) = \frac{1}{6}(r^2 - \frac{ab(a+b)}{r} - a^2 - ab - b^2)$$

8. Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell a<r<b with u=0 on r=a and $\frac{\partial u}{\partial r} = 0$ on r=b。Then let $a \to 0$ in your answer and interpret the result。

$$u(r) = \frac{r^2 - a^2}{6} + \frac{b^3}{3}(\frac{1}{r} - \frac{1}{a})$$

9. A spherical shell with inner radius 1 and outer radius 2 has a steady-state temperature distribution。Its inner boundary is held at $100^0 C$。Its outer boundary satisfies $\frac{\partial u}{\partial r} = -\gamma < 0$，where γ is a constant。

   (a) Find the temperature。(Hint：the temperature depends only on the radius。)
   (b) What are the hottest and coldest temperatures？

   (c) Can you choose γ so that the temperature on its outer boundary is $20^0 C$？

(a) Steady-state means the temperature has stabilized and remains constant over time at every point in the shell。

The steady-state temperature distribution within the spherical shell is determined by solving Laplace's equation in spherical coordinates with radial symmetry。

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{du}{dr}) = 0 \text{，} u(r) = \frac{A}{r} + B$$

r=1，u(1)=100，at r=2， $\dfrac{\partial u}{\partial r} = -\gamma < 0 \Rightarrow A = 4\gamma$

$$u(r) = \dfrac{4\gamma}{r} + 100 - 4\gamma$$

(b) $\because u(r)$ is decreasing。The hottest temperature is u(1)=$100^0 C$，the coldest

    temperature is $u(2) = 100 - 2\gamma^0 C$

(c) $\gamma = 40$ at r=2

10. Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D，with u=g on bdyD by
the energy method。That is，after subtracting two solutions w=u-v，multiply the
Laplace equation for w by w itself and use the divergence theorem。

    (1) Assume two solutions u_1, u_2 and define $w = u_1 - u_2$

        Since $\Delta u_1 = \Delta u_2 = f$，we have $\Delta w = 0$ in D

        On the boundary， $w = u_1 - u_2 = g - g = 0$

    (2) Apply Green first identity

$$\int_D w \Delta w \, dx = 0$$

$$\int_D |\nabla w|^2 \, dx = \int_{\partial D} w \frac{\partial w}{\partial n} \, dS - \int_D w \Delta w \, dx = 0$$

        $|\nabla w|^2 \geq 0 \Rightarrow \nabla w \equiv 0 \Rightarrow w$ is constant，but w=0 on ∂D，implies the constant is

        zero。Therefore $u_1 = u_2$

§

1.   Divergence theorem： $\iint_S \vec{E} \cdot \vec{n} \, dS = \iiint_V div \vec{E} \, dV$

2.   Green first identity： $\int_\Omega (u \Delta v + \nabla u \bullet \nabla v) d\Omega = \int_{\partial\Omega} u \frac{\partial v}{\partial n} \, dS$

1. **Green's First Identity** (used to derive the second identity):

$$\iiint_D \nabla u \cdot \nabla v \, dV = \iint_{\partial D} u \frac{\partial v}{\partial n} dS - \iiint_D u \Delta v \, dV.$$

2. **Second Identity**: Subtract the first identity for u and v swapped:

$$\iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS = \iiint_D (u \Delta v - v \Delta u) \, dV.$$

11. Show that there is no solution of $\Delta u = f$ in D , $\frac{\partial u}{\partial n} = g$ on ∂D in three

dimensions , unless $\iiint_D f dx dy dz = \iint_{\partial D} g dS$. Also show the analogue in one and two

dimensions 。

To demonstrate the necessity of the compatibility condition for the existence of a solution

to the Neumann problem $\Delta u = f$ in D with $\frac{\partial u}{\partial n} = g$ on ∂D , we proceed as follows :

1. Integrate both sides of the Poisson equation over the domain D

$$\iiint_D \Delta u dV = \iiint_D f dV$$

2. Apply the divergence theorem to the left-hand side $\iiint_D \nabla \cdot (\nabla u) dV = \iint_{\partial D} \nabla u \cdot n dS$

Where n is the outward unit normal 。
Substituting the Neumann boundary condition $\nabla u \cdot n = g$

3. Then $\iiint_D f dV = \iint_{\partial D} g dS$

If this equality fails , the assumption that a solution u exists leads to a contradiction 。

12. Check the validity of the maximum principle for the harmonic function

$$u(x, y) = \frac{1 - x^2 - y^2}{1 - 2x + x^2 + y^2} \quad \text{in the disk} \quad \overline{D} = \{x^2 + y^2 \leq 1\}$$

u(x,y) is singular at (1,0) , where it becomes discontinuous 。
The maximum principle requires harmonicity in the open domain and continuity on the
closure 。

Since u fails to be continuous on the closed disk $\overline{D}$, the maximum principle does not

apply 。

13. A function u(x) is subharmonic in D if $\Delta u \geq 0$ in D。
    Prove that its maximum value is attained on ∂D 。(Note that this is not true for the minimum value。)
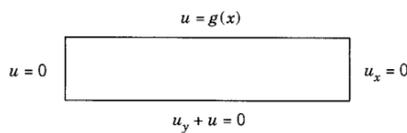
§ 6.2 Rectangles and cubes

$\Delta u = u_{xx} + u_{yy} = 0$ in D。Where D is a rectangle {0<x<a,0<y<b},on each sides one of of the standard boundary conditions is prescribed。(inhomogeneous Dirichlet,Neumann,or Robin)

Examples
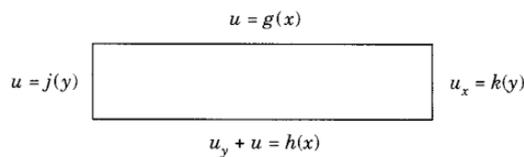
1. Boundary conditions indicates as in the left figure。

$u = g(x)$

$u = j(y)$ ... $u_x = k(y)$

$u_y + u = h(x)$

2. For simplicity,assume h=0,j=0,k=0。

$u = g(x)$

$u = 0$ ... $u_x = 0$

$u_y + u = 0$

We separate the variables

$u(x, y) = X(x) \cdot Y(y)$,then we get

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

Hence there is a constant λ such that $X'' + \lambda X = 0,$ for $0 \leq x \leq a$,$Y'' - \lambda Y = 0$ for $0 \leq y \leq b$

$X'' = -\lambda X$ with $x(0) = X'(a) = 0$

$Y'' = \lambda Y$ with $Y'(0) + Y(0) = 0$

3. …

Exerises

1. Solve $u_{xx} + u_{yy} = 0$ in the rectangle 0<x<a,0<y<b with the following boundary conditions：

   $u_x = -a$ on x=0,$u_x = 0$ on x=a；$u_y = b$ on y=0,$u_y = 0$ on y=b

   $$u(x, y) = \frac{1}{2}x^2 - ax - \frac{1}{2}y^2 + by + c$$

2. Find the harmonic function u(x,y) in the square $D = \{0 < x < \pi, 0 < y < \pi\}$ with

   the boundary conditions：$u_y = 0$ for y=0 and for $y = \pi$

   u=0 for x=0 and $u = \cos^2 y = \frac{1}{2}(1 + \cos 2y)$ for $x = \pi$

   Assume $u(x, y) = X(x)Y(y)$

   $$\nabla^2 u = 0 \Rightarrow \begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$$

   Boundary condition：$Y'(0) = Y'(\pi) = 0$

   $$Y'' + \lambda Y = 0 \Rightarrow Y = A\sin\sqrt{\lambda}\,y + B\cos\sqrt{\lambda}\,y$$

   $$Y'(0) = Y'(\pi) = 0 \Rightarrow \lambda_n = n^2, Y_n(y) = \cos(ny), n = 0, 1, 2, \ldots$$

   $$X'' - n^2 X = 0 \Rightarrow X_n = Ae^{nx} + Be^{-nx}, \ \text{u(0,y)=0} \Rightarrow A + B = 0$$

   Hence $X_n = C_n \sinh(nx)$ (except n=0)

   $$u(x, y) = \frac{A_0 x}{2} + \sum_{n=1}^{\infty} C_n \sinh(nx)\cos(ny)$$

   Apply boundary condition at $x = \pi$，

   $$\frac{A_0 \pi}{2} + \sum_{n=1}^{\infty} C_n \sinh(n\pi)\cos(ny) = \frac{1}{2}(1 + \cos 2y) \Rightarrow A_0 = \frac{1}{\pi}, C_2 = \frac{1}{2\sinh(2\pi)}$$

   $$u(x, y) = \frac{x}{2\pi} + \frac{\sinh(2x)\cos 2y}{2\sinh(2\pi)}$$

3. Find the harmonic function in the square {0<x<1，0<y<1}with the boundary

   conditions $u(x,0) = x, u(x,1) = 0, u_x(0, y) = 0, u_x(1, y) = y^2$

   Assume u(x,y)=v(x,y)+w(x,y)
   v(x,0)=x,v(x,1)=0,v(0,y)=v(1,y)=0

   $w_x(0, y) = 0, w_x(1, y) = y^2$，w(x,0)=w(x,1)=0

   分別解 v，w 然後相加。

   其中用變數分離法解 v(x,y)，得 $v(x, y) = x - \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x)\sinh(n\pi y)$

   $w(x, y) = \sum_{n=1}^{\infty} C_n \cosh(n\pi x)\sinh(n\pi y)$，其中 $C_n = \frac{2(-1)^n}{\pi^3 n^3 \sinh(n\pi)}$

4. Solve the following Neumann problem in the cube {0<x<1，0<y<1，0<z<1}：

$\Delta u = 0$ with $u_z(x, y, 1) = g(x, y)$ and homogeneous Neumann conditions on the

other five faces，where g(x,y) is an arbitrary function with zero average。

$$u(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{-4^{1-\delta_{n0}}}{m\pi\sqrt{m^2+n^2}\sinh(\pi\sqrt{m^2+n^2})} \left(\int_0^1 \int_0^1 g(x',y')\cos(m\pi x')\cos(n\pi y') \right.$$

$$\left. dx'dy' \right) \cos(m\pi x)\cos(n\pi y)\cosh(\pi\sqrt{m^2+n^2}z)$$

Where $\delta_{n0} = 1$ if n=0，and 0 otherwise。

DeepSeek 給出一個例子 $g(x, y) = \cos(\pi x)\cos(\pi y)$，zero average 的意思是：

$\int_0^1 \int_0^1 \cos(\pi x)\cos(\pi y)dxdy = 0$，是 Fourier series 解中的基本函數 $\cos(m\pi x)\cos(n\pi y)$

上面恐怖的係數變成 0，取 $g(x, y) = \cos(m\pi x)\cos(n\pi y), m \geq 1, n \geq 0$

取 m=2，n=1， $g(x, y) = \cos(2\pi x)\cos(\pi y)$ 則此 Neumann 問題的解為：

$$u(x, y, z) = \frac{-1}{2\pi\sqrt{5}\sinh(\pi\sqrt{5})}\cos(2\pi x)\cos(\pi y)\cosh(\pi\sqrt{5}z)$$

5. Find the harmonic function in the semi-infinite strip $\{0 \leq x \leq \pi, 0 \leq y < \infty\}$ that

satisfies the boundary condition： $u(0, y) = u(\pi, y) = 0, u(x,0) = h(x), \lim_{y\to\infty} u(x, y) = 0$

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin(nx)e^{-ny} ， B_n = \frac{2}{\pi}\int_0^\pi h(x')\sin(nx')dx'$$

$$u(x,y) = \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\int_0^\pi h(x')\sin(nx')\,dx' \right) \sin(nx)e^{-ny}$$

What would go awry(歪曲) if we omitted the condition at infinity？

§ 6.3 Poisson formula

A much more interesting case is the Dirichlet problem for a circle。

The rotational invariance of Δ provides a hint that the circle is a natural shape for harmonic functions。

Example

$\Delta u(x, y) = u_{xx} + u_{yy} = 0$ for $x^2 + y^2 < a^2$

$u = h(\theta)$ for $x^2 + y^2 = a^2$

Saparate variables in polar coordinates：

$$u = R(r)\Theta(\theta)，u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0，\frac{r^2 R'' + rR'}{-R} = \frac{\Theta''}{\Theta} = -\lambda$$

With BC： $\Theta(\theta + 2\pi) = \Theta(\theta)$ for $-\infty < \theta < \infty$

Thus， $\lambda = n^2$ and $\Theta(\theta) = A\cos n\theta + B\sin n\theta$ n=1,2,3,···

There is also the solution $\lambda = 0$ with $\Theta(\theta) = A$

$$r^2 R'' + rR' - \lambda R = 0$$

$$R = r^\alpha, R' = \alpha r^{\alpha-1}, R'' = \alpha(\alpha-1)r^{\alpha-2}$$

$$\alpha(\alpha-1) + \alpha - n^2 = 0，\alpha = \pm n \Rightarrow R = Cr^n + Dr^{-n}\ \ (因為\ r=0\ 所以排除\ r^{-n}$$

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$$

···經過一番魔幻步驟，最後得到 Poisson formula：

$$u(r,\theta) = (a^2 - r^2)\int_0^{2\pi} \frac{h(\phi)}{a^2 - 2ar\cos(\theta - \phi) + r^2}\frac{d\phi}{2\pi}.$$

Exercises

1. Suppose that u is a harmonic function in the disk D={r<2} and that $u = 3\sin 2\theta + 1$ for r=2。Without finding the solution，answer the following questions

   (a) Find the maximum value of u in $\overline{D}$

   (b) Calculus the value of u at the origin。

   $$u(r,\theta) = 1 + \frac{3r^2}{4}\sin \theta$$

2. Solve $u_{xx} + u_{yy} = 0$ in the disk {r<a} with the boundary condition $u = 1 + 3\sin\theta$ on

   r=a

   $$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$$

   邊界條件為 $u(a,\theta) = 1 + 3\sin\theta$ ，展開傳立葉級數

   $$1 + 3\sin\theta = A_0 + \sum_{n=1}^{\infty} a^n(A_n\cos n\theta + B_n\sin n\theta)\cdots$$
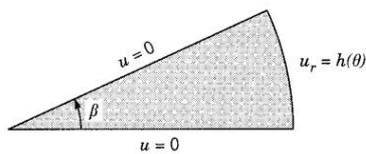
$$u(r,\theta) = 1 + \frac{3r}{a}\sin\theta$$

3. Solve $u_{xx} + u_{yy} = 0$ in the disk {r<a} with the boundary condition $u = \sin^3\theta$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$u(r,\theta) = (\frac{3r}{4a})\sin\theta - (\frac{r^3}{4a^3})\sin 3\theta$$

§ 6.4 circles，wedges，and annuli

Example 1 The wedge

$\{0 < \theta < \beta, 0 < r < a\}$

Boundary condition：

$$u(r,0) = u(r,\beta) = 0, \frac{\partial u}{\partial r}(a,\theta) = h(\theta)$$

$r^2 R'' + rR' - \lambda R = 0$ and $\Theta'' + \lambda\Theta = 0$(變數分離法對圓的結果依然可用)

Now $\Theta(0) = \Theta(\beta) = 0$

這是標準的 eigenvalue problem，其解為 $\lambda = (\frac{n\pi}{\beta})^2, \Theta(\theta) = \sin\frac{n\pi\theta}{\beta}$

$r^2 R'' + rR' - \lambda R = 0$是一個 ODE，解為 $R(r) = r^\alpha$，$\alpha^2 - \lambda = 0$，$\alpha = \sqrt{\lambda} = \frac{n\pi}{\beta}$

(在原點 $r^{-\frac{n\pi}{\beta}} \to \infty$，所以排除。)$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{\frac{n\pi}{\beta}}\sin\frac{n\pi\theta}{\beta}$

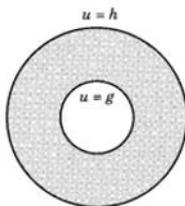Finally，the inhomogeneous boundary condition requires that

$$h(\theta) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{\beta} a^{-1+n\pi/\beta}\sin\frac{n\pi\theta}{\beta}$$

This is just a Fourier sine series in the interval $[0,\beta]$，so

$$A_n = a^{1-n\pi/\beta}\frac{2}{n\pi}\int_0^\beta h(\theta)\sin\frac{n\pi\theta}{\beta}d\theta$$

以下暫略

The Annulus

The exterior of a circle

Exercises

1. Solve $u_{xx} + u_{yy} = 0$ in the the exterior $\{r > a\}$ of a disk，with the boundary condition

    $u = 1 + 3\sin\theta$ on r=a，and the condition at infinity that u be bounded as $r \to \infty$

2. Solve $u_{xx} + u_{yy} = 0$ in the disk r<a with the boundary condition $\dfrac{\partial u}{\partial r} - hu = f(\theta)$，

    where $f(\theta)$ is an arbitary function。Write the answer in terms of the Fourier coefficients of $f(\theta)$。

3. Find the steady-state temperature distribution inside an annular plate $\{1<r<2\}$，whose outer edge (r=2) is insulated，and on whose inner edge (r=1) the temperature is maintained as $\sin^2\theta$。

4. Find the harmonic function u in the semidisk $\{r < 1, 0 < \theta < \pi\}$ with u vanishing on the diameter $(\theta = 0, \pi)$ and $u = \pi\sin\theta - \sin 2\theta$ on r=1

5.

8. An annular plate with inner radius a and outer radius b is held at temperature B at its outer boundary and satisfies the boundary condition $\partial u / \partial r = A$ at its inner boundary, where A and B are constants. Find the temperature if it is at a steady state. (*Hint:* It satisfies the two-dimensional Laplace equation and depends only on r.)

9. Solve $u_{xx} + u_{yy} = 0$ in the wedge $r < a, 0 < \theta < \beta$ with the BCs

    $u = \theta$ on $r = a$, $u = 0$ on $\theta = 0$, and $u = \beta$ on $\theta = \beta$.

    (*Hint:* Look for a function independent of r.)

10. Solve $u_{xx} + u_{yy} = 0$ in the quarter-disk $\{x^2 + y^2 < a^2, x > 0, y > 0\}$ with the following BCs:

    $u = 0$ on $x = 0$ and on $y = 0$ and $\dfrac{\partial u}{\partial r} = 1$ on $r = a$.

    Write the answer as an infinite series and write the first two nonzero terms explicitly.

11. Prove the uniqueness of the Robin problem

$$\Delta u = f \quad \text{in } D, \qquad \frac{\partial u}{\partial n} + au = h \quad \text{on bdy } D,$$

where D is any domain in three dimensions and where a is a positive constant.

12. (a) Prove the following still stronger form of the maximum principle, called the Hopf form of the maximum principle. If $u(\mathbf{x})$ is a non-constant harmonic function in a connected plane domain D with a smooth boundary that has a maximum at $\mathbf{x}_0$ (necessarily on the boundary by the strong maximum principle), then $\partial u / \partial n > 0$ at $\mathbf{x}_0$ where $\mathbf{n}$ is the unit *outward* normal vector. (This is difficult: see [PW] or [Ev].)

    (b) Use part (a) to deduce the uniqueness of the Neumann problem in a connected domain, up to constants.

13. Solve $u_{xx} + u_{yy} = 0$ in the region $\{\alpha < \theta < \beta, a < r < b\}$ with the boundary conditions $u = 0$ on the two sides $\theta = \alpha$ and $\theta = \beta$, $u = g(\theta)$ on the arc $r = a$, and $u = h(\theta)$ on the arc $r = b$.

習作

1. True or false：

    (a) Every harmonic polynomial is homogeneous。　　反例 $f(x, y) = x^2 - y^2 + x$

    (b) Every homogeneous polynomial is harmonic。　　反例 $f(x, y) = x^2$

2. Let A be a constant nonsingular 3×3 matrix，u(x) a C^1 scalar field，and v(x) a C^1 vector field。Set $U(x) = u(Ax)$ and $V(x) = v(Ax)$。Prove that

    (a) $\nabla U(x) = A^T \nabla u(Ax)$

    (b) $\nabla \bullet V(x) = w(Ax)$ where $w(x) = \nabla \bullet (Av)(x)$

3. The Weak Maximum Principle：Let $\Omega \subset R^2$ be a bounded domain。
    Let u(x,y,z) solve the Poisson equation $-\Delta u = f(x, y, z)$，where f(x,y,z)<0 for all $(x,y,z) \in \Omega$。
    (a) Prove that the maximum value of u ocurs on the boundary $\partial \Omega$。
    (b) Generalize your result to the case $f(x, y, z) \leq 0$

4.