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Ch6 Generalized Functions and Green's Functions

6.1 Generalized Functions

6.1.1. Evaluate the following integrals: (a) $\int_{-\pi}^{\pi} \delta(x) \cos x \, dx$, (b) $\int_{1}^{2} \delta(x)\,(x-2)\,dx$, (c) $\int_0^3 \delta_1(x)\, e^x \, dx$, (d) $\int_1^e \delta(x-2)\, \log x \, dx$, (e) $\int_0^1 \delta\left(x - \frac{1}{3}\right) x^2 \, dx$, (f) $\int_{-1}^{1} \frac{\delta(x+2)\,dx}{1+x^2}$.

6.1.2. Simplify the following generalized functions; then write out how they act on a suitable test function $u(x)$: (a) $e^x\,\delta(x)$, (b) $x\,\delta(x-1)$, (c) $3\,\delta_1(x) - 3x\,\delta_{-1}(x)$, (d) $\dfrac{\delta(x-1)}{x+1}$, (e) $(\cos x)\left[\,\delta(x) + \delta(x-\pi) + \delta(x+\pi)\,\right]$, (f) $\dfrac{\delta_1(x) - \delta_2(x)}{x^2+1}$.

6.1.3. Define the generalized function $\varphi(x) = \delta(x+1) - \delta(x-1)$: (a) as a limit of ordinary functions; (b) using duality.

6.1.4. Find and sketch a graph of the derivative (in the context of generalized functions) of the following functions:

(a) $f(x) = \begin{cases} x^2, & 0 < x < 3, \\ x, & -1 < x < 0, \\ 0, & \text{otherwise}, \end{cases}$ (b) $g(x) = \begin{cases} \sin|\,x\,|, & |\,x\,| < \frac{1}{2}\pi, \\ 0, & \text{otherwise}, \end{cases}$

(c) $h(x) = \begin{cases} \sin \pi x, & x > 1, \\ 1 - x^2, & -1 < x < 1, \\ e^x, & x < -1, \end{cases}$ (d) $k(x) = \begin{cases} \sin x, & x < -\pi, \\ x^2 - \pi^2, & -\pi < x < 0, \\ e^{-x}, & x > 0. \end{cases}$

6.1.5. Find the first and second derivatives of the functions (a) $f(x) = \begin{cases} x+1, & -1 < x < 0, \\ 1 - x, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases}$

(b) $k(x) = \begin{cases} |\,x\,|, & -2 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$ (c) $s(x) = \begin{cases} 1 + \cos \pi x, & -1 < x < 1, \\ 0, & \text{otherwise}. \end{cases}$

6.1.6. Find the first and second derivatives of $f(x) =$ (a) $e^{-|\,x\,|}$, (b) $2\,|\,x\,| - |\,x-1\,|$, (c) $|\,x^2 + x\,|$, (d) $x \,\operatorname{sign}(x^2 - 4)$, (e) $\sin|\,x\,|$, (f) $|\sin x\,|$, (g) $\operatorname{sign}(\sin x)$.

6.1.7. Explain why the Gaussian functions $g_n(x) = \dfrac{n}{\sqrt{\pi}}\, e^{-n^2 x^2}$ have the delta function $\delta(x)$ as their limit as $n \to \infty$.

6.1.8. In this exercise, we realize the delta function $\delta_\xi(x)$ as a limit of functions on a finite interval $[a,b]$. Let $a < \xi < b$.

(a) Prove that the functions $\widetilde{g}_n(x) = \dfrac{g_n(x-\xi)}{M_n}$, where $g_n(x)$ is given by (6.10) and $M_n = \int_a^b g_n(x-\xi)\,dx$, satisfy (6.8–9), and hence $\lim_{n\to\infty} \widetilde{g}_n(x) = \delta_\xi(x)$.

(b) One can, alternatively, relax the second condition (6.9) to $\lim_{n\to\infty} \int_a^b g_n(x-\xi)\,dx = 1$. Show that, under this relaxed definition, $\lim_{n\to\infty} g_n(x-\xi) = \delta_\xi(x)$.

6.1.9. For each positive integer n, let $g_n(x) = \begin{cases} \frac{1}{2}n, & |x| < 1/n, \\ 0, & \text{otherwise.} \end{cases}$ (a) Sketch a graph of $g_n(x)$. (b) Show that $\lim_{n \to \infty} g_n(x) = \delta(x)$. (c) Evaluate $f_n(x) = \int_{-\infty}^{x} g_n(y)\,dy$ and sketch a graph. Does the sequence $f_n(x)$ converge to the step function $\sigma(x)$ as $n \to \infty$? (d) Find the derivative $h_n(x) = g_n'(x)$. (e) Does the sequence $h_n(x)$ converge to $\delta'(x)$ as $n \to \infty$?

6.1.10. Answer Exercise 6.1.9 for the *hat functions* $g_n(x) = \begin{cases} n - n^2|x|, & |x| < 1/n, \\ 0, & \text{otherwise.} \end{cases}$

6.1.11. Justify the formula $x\,\delta(x) = 0$ using (a) limits, (b) duality.

6.1.12. (a) Justify the formula $\delta(2x) = \frac{1}{2}\,\delta(x)$ by (i) limits, (ii) duality. (b) Find a similar formula for $\delta(ax)$ when $a > 0$. (c) What about when $a < 0$?

6.1.13. (a) Prove that $\sigma(\lambda x) = \sigma(x)$ for any $\lambda > 0$. (b) What about if $\lambda < 0$? (c) Use parts (a,b) to deduce that $\delta(\lambda x) = \dfrac{1}{|\lambda|}\,\delta(x)$ for any $\lambda \neq 0$.

6.1.14. Let $g(x)$ be a continuously differentiable function with $g'(x) \neq 0$ for all $x \in \mathbb{R}$. Does the composition $\delta(g(x))$ make sense as a distribution? If so, can you identify it?

6.1.15. Let $\xi < a$. Sketch the graphs of (a) $s(x) = \int_a^x \delta_\xi(z)\,dz$, (b) $r(x) = \int_a^x \sigma_\xi(z)\,dz$.

6.1.16. Justify the formula $\lim_{n \to \infty} n\left[\delta\left(x - \frac{1}{n}\right) - \delta\left(x + \frac{1}{n}\right)\right] = -2\,\delta'(x)$.

6.1.17. Define the generalized function $\delta''(x)$:
(a) as a limit of ordinary functions; (b) using duality.

6.1.18. Let $\delta_\xi^{(k)}(x)$ denote the k^{th} derivative of the delta function $\delta_\xi(x)$. Justify the formula $\langle \delta_\xi^{(k)}, u \rangle = (-1)^k\,u^{(k)}(\xi)$ whenever $u \in \mathrm{C}^k$ is k–times continuously differentiable.

6.1.19. According to (6.22), $x\,\delta(x) = 0$. On the other hand, by Leibniz' rule, $(x\,\delta(x))' = \delta(x) + x\,\delta'(x)$ is apparently not zero. Can you explain this paradox?

6.1.20. If $f \in \mathrm{C}^1$, should $(f\,\delta)' = f\,\delta'$ or $f'\,\delta + f\,\delta'$?

6.1.21. (a) Use duality to justify the formula $f(x)\,\delta'(x) = f(0)\,\delta'(x) - f'(0)\,\delta(x)$ when $f \in \mathrm{C}^1$.
(b) Find a similar formula for $f(x)\,\delta^{(n)}(x)$ as the product of a sufficiently smooth function and the n^{th} derivative of the delta function.

6.1.22. Use Exercise 6.1.21 to simplify the following generalized functions; then write out how they act on a suitable test function $u(x)$:
(a) $\varphi(x) = (x-2)\,\delta'(x)$, (b) $\psi(x) = (1+\sin x)\big[\,\delta(x) + \delta'(x)\,\big]$,
(c) $\chi(x) = x^2\big[\,\delta(x-1) - \delta'(x-2)\,\big]$, (d) $\omega(x) = e^x\,\delta''(x+1)$.

6.1.23. Prove that if $f(x)$ is a continuous function, and $\int_a^b f(x)\,dx = 0$ for every interval $[a,b]$, then $f(x) \equiv 0$ everywhere.

6.1.24. Write out a rigorous proof that there is no continuous function $\delta_\xi(x)$ such that the inner product identity (6.20) holds for every continuous function $u(x)$.

6.1.25. *True or false*: The sequence (6.24) converges uniformly.

6.1.26. *True or false*: $\|\,\delta\,\| = 1$.

The Fourier Series and Delta Functions

6.1.27. Determine the real and complex Fourier series for $\delta(x-\xi)$, where $-\pi < \xi < \pi$. What periodic generalized function(s) do they represent?

6.1.28. Determine the Fourier sine series and the Fourier cosine series for $\delta(x-\xi)$, where $0 < \xi < \pi$. Which periodic generalized functions do they represent?

6.1.29. Let $n > 0$ be a positive integer. (*a*) For integers $0 \le j < n$, find the complex Fourier series of the 2π–periodically extended delta functions $\widetilde{\delta}_j(x) = \widetilde{\delta}(x - 2j\pi/n)$. (*b*) Prove that their Fourier coefficients satisfy the periodicity condition $c_k = c_l$ whenever $k \equiv l \mod n$. (*c*) Conversely, given complex Fourier coefficients that satisfy the periodicity condition $c_k = c_l$ whenever $k \equiv l \mod n$, prove that the corresponding Fourier series represents a linear combination of the preceding periodically extended delta functions $\widetilde{\delta}_0(x), \ldots, \widetilde{\delta}_{n-1}(x)$. *Hint*: Use Example B.22. (*d*) Prove that a complex Fourier series represents a 2π–periodic function that is constant on the subintervals $2\pi j/n < x < 2\pi(j+1)/n$, for $j \in \mathbb{Z}$, if and only if its Fourier coefficients satisfy the conditions

$$k\,c_k = l\,c_l, \qquad k \equiv l \not\equiv 0 \mod n, \qquad\qquad c_k = 0, \qquad 0 \ne k \equiv 0 \mod n.$$

6.1.30. (*a*) Find the complex Fourier series for the derivative of the delta function $\delta'(x)$ by direct evaluation of the coefficient formulas. (*b*) Verify that your series can be obtained by term-by-term differentiation of the series for $\delta(x)$. (*c*) Write a formula for the n^{th} partial sum of your series. (*d*) Use a computer graphics package to investigate the convergence of the series.

6.1.31. What is the Fourier series for the generalized function $g(x) = x\,\delta(x)$? Can you obtain this result through multiplication of the individual Fourier series (3.37), (6.37)?

6.1.32. Apply the method of Exercise 3.2.59 to find the complex Fourier series for the function $f(x) = \delta(x)\,e^{\,i\,x}$. Which Fourier series do you get? Can you explain what is going on?

6.1.33. In Exercise 6.1.12 we established the identity $\delta(x) = 2\,\delta(2x)$. Does this hold on the level of Fourier series? Can you explain why or why not?

6.1.34. How should one interpret the formula (6.38) for the periodic extension of the delta function (*a*) as a limit? (*b*) as a linear functional?

6.1.35. Write down the complex Fourier series for e^x. Differentiate term by term. Do you get the same series? Explain your answer.

6.1.36. *True or false*: If you integrate the Fourier series for the delta function $\delta(x)$ term by term, you obtain the Fourier series for the step function $\sigma(x)$.

6.1.37. Find the Fourier series for the function $\delta(x)$ on the interval $-1 \le x \le 1$. Which (generalized) function does the Fourier series represent?

6.1.38.. Prove that $\cos n x \rightharpoonup 0$ (weakly) as $n \to \infty$ on any bounded interval $[a, b]$.

6.1.39. Prove that if $u_n \to u$ in norm, then $u_n \rightharpoonup u$ weakly.

6.1.40. *True or false*: (*a*) If $u_n \to u$ uniformly on $[a,b]$, then $u_n \rightharpoonup u$ weakly.
(*b*) If $u_n(x) \to u(x)$ pointwise, then $u_n \rightharpoonup u$ weakly.

6.1.41. Prove that the sequence $f_n(x) = \cos^2 nx$ converges weakly on $[-\pi,\pi]$. What is the limiting function?

6.1.42. Answer Exercise 6.1.41 when $f_n(x) = \cos^3 nx$.

6.1.43. Discuss the weak convergence of the Fourier series for the derivative $\delta'(x)$ of the delta function.

6.2 Green's Functions for one-dimensional BVP

6.2.1. Let $c > 0$. Find the Green's function for the boundary value problem $-cu'' = f(x)$, $u(0) = 0$, $u'(1) = 0$, which models the displacement of a uniform bar of unit length with one fixed and one free end under an external force. Then use superposition to write down a formula for the solution. Verify that your integral formula is correct by direct differentiation and substitution into the differential equation and boundary conditions.

6.2.2. A uniform bar of length $\ell = 4$ has constant stiffness $c = 2$. Find the Green's function for the case that (*a*) both ends are fixed; (*b*) one end is fixed and the other is free. (*c*) Why is there no Green's function when both ends are free?

6.2.3. A point 2 cm along a 10 cm bar experiences a displacement of 1 mm under a concentrated force of 2 newtons applied at the midpoint of the bar. How far does the midpoint deflect when a concentrated force of 1 newton is applied at the point 2 cm along the bar?

6.2.4. The boundary value problem $-\dfrac{d}{dx}\left(c(x)\dfrac{du}{dx}\right) = f(x)$, $u(0) = u(1) = 0$, models the displacement $u(x)$ of a nonuniform elastic bar with stiffness $c(x) = \dfrac{1}{1+x^2}$ for $0 \le x \le 1$. (*a*) Find the displacement when the bar is subjected to a constant external force, $f \equiv 1$. (*b*) Find the Green's function for the boundary value problem. (*c*) Use the resulting superposition formula to check your solution to part (*a*). (*d*) Which point $0 < \xi < 1$ on the bar is the "weakest", i.e., the bar experiences the largest displacement under a unit impulse concentrated at that point?

6.2.5. Answer Exercise 6.2.4 when $c(x) = 1 + x$.

6.2.6. Consider the boundary value problem $-u'' = f(x),\ \ u(0) = 0,\ \ u(1) = 2u'(1)$.
(a) Find the Green's function. (b) Which of the fundamental properties does your Green's function satisfy? (c) Write down an explicit integral formula for the solution to the boundary value problem, and prove its validity by a direct computation. (d) Explain why the related boundary value problem $-u'' = f,\ u(0) = 0,\ u(1) = u'(1)$, does not have a Green's function.

6.2.7. For n a positive integer, set $f_n(x) = \begin{cases} \frac{1}{2}n, & |x-\xi| < \frac{1}{n}, \\ 0, & \text{otherwise.} \end{cases}$

(a) Find the solution $u_n(x)$ to the boundary value problem $-u'' = f_n(x)$, $u(0) = u(1) = 0$, assuming $0 < \xi - \frac{1}{n} < \xi + \frac{1}{n} < 1$. (b) Prove that $\lim_{n\to\infty} u_n(x) = G(x;\xi)$ converges to the Green's function (6.51). Why should this be the case? (c) Reconfirm the result in part (b) by graphing $u_5(x), u_{15}(x), u_{25}(x)$, along with $G(x;\xi)$ when $\xi = .3$.

6.2.8. Solve the boundary value problem $-4u'' + 9u = 0,\ \ u(0) = 0,\ \ u(2) = 1$. Is your solution unique?

6.2.9. *True or false*: The Neumann boundary value problem $-u'' + u = 1,\ \ u'(0) = u'(1) = 0$, has a unique solution.

6.2.10. Use the Green's function (6.64) to solve the boundary value problem (6.57) when the forcing function is $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \le 1. \end{cases}$

6.2.11. Let $\omega > 0$. (a) Find the Green's function for the mixed boundary value problem

$$-u'' + \omega^2 u = f(x),\ \ u(0) = 0,\ \ u'(1) = 0.$$

(b) Use your Green's function to find the solution when $f(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} < x \le 1. \end{cases}$

6.2.12. Suppose $\omega > 0$. Does the Neumann boundary value problem $-u'' + \omega^2 u = f(x)$, $u'(0) = u'(1) = 0$ admit a Green's function? If not, explain why not. If so, find it, and then write down an integral formula for the solution of the boundary value problem.

6.2.13. (a) Prove the addition formula (6.63) for the hyperbolic sine function.
(b) Find the corresponding addition formula for the hyperbolic cosine.

6.2.14. Prove the differentiation formula (6.55).

6.3 Green's Functions for the Planar Poisson Equations

6.3.1. Let C_R be a circle of radius R centered at the origin and $\mathbf{n}$ its unit outward normal. Let $f(r,\theta)$ be a function expressed in polar coordinates. Prove that $\partial f/\partial \mathbf{n} = \partial f/\partial r$ on C_R.

6.3.2. Let $f(x) > 0$ be a continuous, positive function on the interval $a \le x \le b$. Let Ω be the domain lying between the graph of $f(x)$ on the interval $[a,b]$ and the x–axis. Explain why (6.77) reduces to the usual calculus formula for the area under the graph of f.

6.3.3. Explain what happens to the conclusion of Lemma 6.16 if Ω is not a connected domain.

6.3.4. Can you find constants c_n such that the functions $g_n(x,y) = c_n[1+n^2(x^2+y^2)]^{-1}$ converge to the two-dimensional delta function: $g_n(x,y) \to \delta(x,y)$ as $n \to \infty$?

6.3.5. Explain why the two-dimensional delta function satisfies the scaling law

$$\delta(\beta\,x, \beta\,y) = \frac{1}{\beta^2}\,\delta(x,y), \qquad \text{for} \qquad \beta > 0.$$

6.3.6. Write out a polar coordinate formula, in terms of $\delta(r - r_0)$ and $\delta(\theta - \theta_0)$, for the two-dimensional delta function $\delta(x - x_0, y - y_0) = \delta(x - x_0)\,\delta(y - y_0)$.

6.3.7. *True or false*: $\delta(\mathbf{x}) = \delta(\|\,\mathbf{x}\,\|)$.

6.3.8. Suppose that $\xi = f(x,y)$, $\eta = g(x,y)$ defines a one-to-one C^1 map from a domain $D \subset \mathbb{R}^2$ to the domain $\Omega = \{\,(\xi,\eta) = (f(x,y), g(x,y))\,|\,(x,y) \in D\,\} \subset \mathbb{R}^2$, and has nonzero Jacobian determinant: $J(x,y) = f_x g_y - f_y g_x \neq 0$ for all $(x,y) \in D$. Suppose further that $(0,0) = (f(x_0,y_0), g(x_0,y_0)) \in \Omega$ for $(x_0,y_0) \in D$. Prove the following formula governing the effect of the map on the two-dimensional delta function:

$$\delta(f(x,y), g(x,y)) = \frac{\delta(x - x_0, y - y_0)}{|\,J(x_0,y_0)\,|}\,. \tag{6.126}$$

6.3.9. Suppose $f(x,y) = \begin{cases} 1, & 3x - 2y > 1, \\ 0, & 3x - 2y < 1. \end{cases}$ Compute its partial derivatives $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ in the sense of generalized functions.

6.3.10. Find a series solution to the rectangular boundary value problem (4.91–92) when the boundary data $f(x) = \delta(x - \xi)$ is a delta function at a point $0 < \xi < a$. Is your solution infinitely differentiable inside the rectangle?

6.3.11. Answer Exercise 6.3.10 when $f(x) = \delta'(x - \xi)$ is the derivative of the delta function.

6.3.12. A 1 meter square plate is subject to the Neumann boundary conditions $\partial u/\partial \mathbf{n} = 1$ on its entire boundary. What is the equilibrium temperature? Explain.

6.3.13. A *conservation law* for an equilibrium system in two dimensions is, by definition, a divergence expression

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0 \qquad (6.127)$$

that vanishes for all solutions.

(a) Given a conservation law prescribed by $\mathbf{v} = (X, Y)$ defined on a simply connected domain D, show that the line integral $\int_C \mathbf{v} \cdot \mathbf{n}\, ds = \int_C X\, dy - Y\, dx$ is path-independent, meaning that its value depends only on the endpoints of the curve C.

(b) Show that the Laplace equation can be written as a conservation law, and write down the corresponding path-independent line integral.

Note: Path-independent integrals are of importance in the study of cracks, dislocations, and other material singularities, [**49**].

6.3.14. In two-dimensional dynamics, a *conservation law* is an equation of the form

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} = 0, \qquad (6.128)$$

in which T is the *conserved density*, while $\mathbf{v} = (X, Y)$ represents the associated *flux*.

(a) Prove that, on a bounded domain $\Omega \subset \mathbb{R}^2$, the rate of change of the integral $\iint_\Omega T\, dx\, dy$ of the conserved density depends only on the flux through the boundary $\partial\Omega$.

(b) Write the partial differential equation $u_t + u\,u_x + u\,u_y = 0$ as a conservation law. What is the integrated version?

6.3.15. A circular disk of radius 1 is subject to a heat source of unit magnitude on the subdisk $r \leq \frac{1}{2}$. Its boundary is kept at 0°.

(a) Write down an integral formula for the equilibrium temperature.

(b) Use radial symmetry to find an explicit formula for the equilibrium temperature.

6.3.16. A circular disk of radius 1 meter is subject to a unit concentrated heat source at its center and has completely insulated boundary. What is the equilibrium temperature?

6.3.17. (a) For $n > 0$, find the solution to the boundary value problem

$$-\Delta u = \frac{n}{\pi}\, e^{-n\,(x^2+y^2)}, \qquad x^2 + y^2 < 1, \qquad u(x,y) = 0, \qquad x^2 + y^2 = 1.$$

(b) Discuss what happens in the limit as $n \to \infty$.

6.3.18. (a) Use the Method of Images to construct the Green's function for a half-plane $\{y > 0\}$ that is subject to homogeneous Dirichlet boundary conditions. *Hint*: The image point is obtained by reflection. (b) Use your Green's function to solve the boundary value problem

$$-\Delta u = \frac{1}{1+y}, \qquad y > 0, \qquad u(x,0) = 0.$$

6.3.19. Construct the Green's function for the half-disk $\Omega = \{x^2 + y^2 < 1,\ y > 0\}$ when subject to homogeneous Dirichlet boundary conditions. *Hint*: Use three image points.

6.3.20. Prove directly that the Poisson kernel (6.137) solves the Laplace equation for all $r < 1$.

6.3.21. Provide the details for the following alternative method for solving the homogeneous Dirichlet boundary value problem for the Poisson equation on the unit square:

$$u_{xx} - u_{yy} = f(x,y), \quad u(x,0) = 0, \quad u(x,1) = 0, \quad u(0,y) = 0, \quad u(1,y) = 0, \quad 0 < x,\ y < 1.$$

(*a*) Write both $u(x,y)$ and $f(x,y)$ as Fourier sine series in y whose coefficients depend on x. (*b*) Substitute these series into the differential equation, and equate Fourier coefficients to obtain an infinite system of ordinary boundary value problems for the x-dependent Fourier coefficients of u. (*c*) Use the Green's functions for each boundary value problem to write out the solution and hence a series for the solution to the original boundary value problem. (*d*) Implement this method for the following forcing functions:
(*i*) $f(x,y) = \sin \pi y$, (*ii*) $f(x,y) = \sin \pi x \, \sin 2\pi y$, (*iii*) $f(x,y) = 1$.

6.3.22. Use the method of Exercise 6.3.21 to find a series representation for the Green's function of a unit square subject to Dirichlet boundary conditions.

6.3.23. Write out the details of how to derive (6.134) from (6.133).

6.3.24. *True or false*: If the gravitational potential at a point **a** is greater than its value at the point **b**, then the magnitude of the gravitational force at **a** is greater than its value at **b**.

6.3.25. (*a*) Write down integral formulas for the gravitational potential and force due to a square plate $S = \{-1 \le x, y \le 1\}$ of unit density $\rho = 1$. (*b*) Use numerical integration to calculate the gravitational force at the points $(2,0)$ and $(\sqrt{2}, \sqrt{2})$. Before starting, try to predict which point experiences the stronger force, and then check your prediction.

6.3.26. An equilateral triangular plate with unit area exerts a gravitational force on an observer sitting a unit distance away from its center. Is the force greater if the observer is located opposite a vertex of the triangle or opposite a side? Is the force greater than or less than that exerted by a circular plate of the same area? Use numerical integration to evaluate the double integrals.

6.3.27. Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the line $-\infty < x < \infty$. Use the d'Alembert formula (2.82) to solve the initial value problem $u(0,x) = \delta(x-a)$, $u_t(0,x) = 0$. Can you realize your solution as the limit of classical solutions?

6.3.28. Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the line $-\infty < x < \infty$. Use the d'Alembert formula (2.82) to solve the initial value problem $u(0,x) = 0$, $u_t(0,x) = \delta(x-a)$, modeling the effect of striking the string with a highly concentrated blow at the point $x = a$. Graph the solution at several times. Discuss the behavior of any discontinuities in the solution. In particular, show that $u(t,x) \neq 0$ on the domain of influence of the point $(a, 0)$.

6.3.29. (a) Write down the solution $u(t,x)$ to the wave equation $u_{tt} = 4u_{xx}$ on the real line with initial data $u(0,x) = \begin{cases} 1-|x|, & |x| \le 1, \\ 0, & \text{otherwise}, \end{cases}$ $\dfrac{\partial u}{\partial t}(0,x) = 0$. (b) Explain why $u(t,x)$ is not a classical solution to the wave equation. (c) Determine the derivatives $\partial^2 u/\partial t^2$ and $\partial^2 u/\partial x^2$ in the sense of distributions (generalized functions) and use this to justify the fact that $u(t,x)$ solves the wave equation in a distributional sense.

6.3.30. A piano string of length $\ell = 3$ and wave speed $c = 2$ with both ends fixed is hit by a hammer $\frac{1}{3}$ of the way along. The initial-boundary value problem that governs the resulting vibrations of the string is

$$\frac{\partial^2 u}{\partial t^2} = 4\,\frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,0) = 0 = u(t,3), \qquad u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = \delta(x-1).$$

(a) What are the fundamental frequencies of vibration?
(b) Write down the solution to the initial-boundary value problem in Fourier series form.
(c) Write down the Fourier series for the velocity $\partial u/\partial t$ of your solution.
(d) Write down the d'Alembert formula for the solution, and sketch a picture of the string at four or five representative times.
(e) *True or false*: The solution is periodic in time. If true, what is the period? If false, explain what happens as t increases.

6.3.31. (a) Write down a Fourier series for the solution to the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,-1) = 0 = u(t,1), \qquad u(0,x) = \delta(x), \qquad \frac{\partial u}{\partial t}(0,x) = 0.$$

(b) Write down an analytic formula for the solution, i.e., sum your series. (c) In what sense does the series solution in part (a) converge to the true solution? Do the partial sums provide a good approximation to the actual solution?

6.3.32. Answer Exercise 6.3.31 for

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}\,, \qquad u(t,-1) = 0 = u(t,1), \qquad u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = \delta(x).$$