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Ch5 Finite Differences

5.1 Finite Difference Approximations

5.1.1. Use the finite difference formula (5.3) with step sizes $h = .1, .01$, and $.001$ to approximate the derivative $u'(1)$ of the following functions $u(x)$. Discuss the accuracy of your approximation. (a) x^4, (b) $\dfrac{1}{1+x^2}$, (c) $\log x$, (d) $\cos x$, (e) $\tan^{-1} x$.

5.1.2. Repeat Exercise 5.1.1 using the centered difference formula (5.6). Compare your approximations with those in the previous exercise — are the values in accordance with the claimed orders of accuracy?

5.1.3. Approximate the second derivative $u''(1)$ of the functions in Exercise 5.1.1 using the finite difference formula (5.5) with $h = .1, .01$, and $.001$. Discuss the accuracy of your approximations.

5.1.4. Construct finite difference approximations to the first and second derivatives of a function $u(x)$ using its values at the points $x-k, x, x+h$, where $h, k \ll 1$ are of comparable size, but not necessarily equal. What can you say about the error in the approximation?

5.1.5. In this exercise, you are asked to derive some basic *one-sided finite difference formulas*, which are used for approximating derivatives of functions at or near the boundary of their domain. (a) Construct a finite difference formula that approximates the derivative $u'(x)$ using the values of $u(x)$ at the points $x, x+h$, and $x+2h$. What is the order of your formula? (b) Find a finite difference formula for $u''(x)$ that involves the same three function values. What is its order? (c) Test your formulas by computing approximations to the first and second derivatives of $u(x) = e^{x^2}$ at $x = 1$ using step sizes $h = .1, .01$, and $.001$. What is the error in your numerical approximations? Are the errors compatible with the theoretical orders of the finite difference formulas? Discuss why or why not. (d) Answer part (c) at the point $x = 0$.

5.1.6. (a) Using the function values $u(x)$, $u(x+h)$, $u(x+3h)$, construct a numerical approximation to the derivative $u'(x)$. (b) What is the order of accuracy of your approximation? (c) Test your approximation on the function $u(x) = \cos x$ at $x = 1$ using the step sizes $h = .1, .01$, and $.001$. Are the errors consistent with your answer in part (b)?

5.1.7. Answer Exercise 5.1.6 for the second derivative $u''(x)$.

5.1.8. (a) Find the order of the five-point centered finite difference approximation

$$u'(x) \approx \frac{-u(x+2h) + 8u(x+h) - 8u(x-h) + u(x-2h)}{12h}.$$

(b) Test your result on the function $(1+x^2)^{-1}$ at $x = 1$ using the values $h = .1, .01, .001$.

5.1.9. (a) Using the formula in Exercise 5.1.8 as a guide, find five-point finite difference formulas to approximate (i) $u''(x)$, (ii) $u'''(x)$, (iii) $u^{(iv)}(x)$. What is the order of accuracy?
(b) Test your formulas on the function $(1+x^2)^{-1}$ at $x = 1$ using the values $h = .1, .01, .001$.

5.2 Numerical Algorithms for the heat eqation

5.2.1. Suppose we seek to approximate the solution to the initial-boundary value problem

$$u_t = 5\,u_{xx}, \qquad u(t,0) = u(t,3) = 0, \qquad u(0,x) = x(x-1)(x-3), \qquad 0 \le x \le 3,$$

by employing the explicit scheme (5.14). (a) Given the spatial mesh size $\Delta x = .1$, what range of time steps Δt can be used to produce an accurate numerical approximation?
(b) Test your prediction by implementing the scheme using one value of Δx in the allowed range and one value outside.

5.2.2. Solve the following initial-boundary value problem

$$u_t = u_{xx}, \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = f(x), \qquad 0 \le x \le 1,$$

with initial data $f(x) = \begin{cases} 2\left|x - \frac{1}{6}\right| - \frac{1}{3}, & 0 \le x \le \frac{1}{3}, \\ 0, & \frac{1}{3} \le x \le \frac{2}{3}, \\ \frac{1}{2} - 3\left|x - \frac{5}{6}\right|, & \frac{2}{3} \le x \le 1, \end{cases}$ using

(i) the explicit scheme (5.14); (ii) the implicit scheme (5.29); and (iii) the Crank–Nicolson scheme (5.32). Use space step sizes $\Delta x = .1$ and $.05$, and suitably chosen time steps Δt. Discuss which features of the solution can be observed in your numerical approximations.

5.2.3. Repeat Exercise 5.2.2 for the initial-boundary value problem $u_t = 3u_{xx}$, $u(0,x) = 0$, $u(t,-1) = 1$, $u(t,1) = -1$, using space step sizes $\Delta x = .2$ and $.1$.

5.2.4. (a) Solve the initial-boundary value problem

$$u_t = u_{xx}, \qquad u(t,-1) = u(t,1) = 0, \qquad u(0,x) = |\,x\,|^{1/2} - x^2, \qquad -1 \le x \le 1,$$

using (i) the explicit scheme (5.14); (ii) the implicit scheme (5.29); (iii) the Crank–Nicolson scheme (5.32). Use $\Delta x = .1$ and an appropriate time step Δt. Compare your numerical solutions at times $t = 0, .01, , .02, .05, .1, .3, .5, 1.0$, and discuss your findings. (b) Repeat

part (a) for the implicit and Crank-Nicolson schemes with $\Delta x = .01$. Why aren't you being asked to implement the explicit scheme?

5.2.5. Use the implicit scheme with spatial mesh sizes $\Delta x = .1$ and $.05$ and appropriately chosen values of the time step Δt to investigate the solution to the periodically forced boundary value problem $u_t = u_{xx}$, $u(0,x) = 0$, $u(t,0) = \sin 5\pi t$, $u(t,1) = \cos 5\pi t$. Is your solution periodic in time?

5.2.6. (a) How would you modify (i) the explicit scheme; (ii) the implicit scheme; to deal with Neumann boundary conditions? *Hint*: Use the one-sided finite difference formulae found in Exercise 5.1.5 to approximate the derivatives at the boundary.
(b) Test your proposals on the boundary value problem

$$u_t = u_{xx}, \qquad u(0,x) = \tfrac{1}{2} + \cos 2\pi x - \tfrac{1}{2}\cos 3\pi x, \qquad u_x(t,0) = 0 = u_x(t,1),$$

using space step sizes $\Delta x = .1$ and $.01$ and appropriate time steps. Compare your numerical solution with the exact solution at times $t = .01, .03, .05$, and explain any discrepancies.

5.2.7. (a) Design an explicit numerical scheme for approximating the solution to the initial-boundary value problem

$$u_t = \gamma\, u_{xx} + s(x), \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = f(x), \qquad 0 \le x \le 1,$$

for the heat equation with a *source term* $s(x)$. (*b*) Test your scheme when

$$\gamma = \tfrac{1}{6}, \qquad s(x) = x(1-x)(10-22x), \qquad f(x) = \begin{cases} 2\left|x - \frac{1}{6}\right| - \frac{1}{3}, & 0 \le x \le \frac{1}{3}, \\ 0, & \frac{1}{3} \le x \le \frac{2}{3}, \\ \frac{1}{2} - 3\left|x - \frac{5}{6}\right|, & \frac{2}{3} \le x \le 1, \end{cases}$$

using space step sizes $\Delta x = .1$ and $.05$, and a suitably chosen time step Δt. Are your two numerical solutions close? (*c*) What is the long-term behavior of the solution? Can you find a formula for its eventual profile? (*d*) Design an implicit scheme for the same problem. Does this affect the behavior of your numerical solution? What are the advantages of the implicit scheme?

5.2.8. Consider the initial-boundary value problem for the *lossy diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha\, u, \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = f(x), \qquad \begin{array}{c} t \ge 0, \\ 0 \le x \le 1, \end{array}$$

where $\alpha > 0$ is a positive constant. (*a*) Devise an explicit finite difference method for computing a numerical appoximation to the solution. (*b*) For what mesh sizes would you expect your method to provide a good approximation to the solution? (*c*) Discuss the case when $\alpha < 0$.

5.2.9. Consider the initial-boundary value problem for the *diffusive transport equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\,\frac{\partial u}{\partial x}, \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = x(1-x), \qquad \begin{array}{c} t \ge 0, \\ 0 \le x \le 1. \end{array}$$

(*a*) Devise an explicit finite difference scheme for computing numerical appoximations to the solution. *Hint*: Make sure your approximations are of comparable order. (*b*) For what range of time step sizes would you expect your method to provide a decent approximation to the solution? (*c*) Test your answer in part (*b*) for the spatial step size $\Delta x = .1$.

5.2.10. (a) Show that using the centered difference approximation (5.6) to approximate the time derivative leads to *Richardson's method* for numerically solving the heat equation:

$$u_{j+1,m} = u_{j-1,m} + 2\mu\,(u_{j,m+1} - 2\,u_{j,m} + u_{j,m-1}), \qquad \begin{array}{c} j = 1, 2, \ldots\,, \\ m = 1, \ldots, n-1, \end{array}$$

where $\mu = \gamma\,\Delta t/(\Delta x)^2$ is as in (5.15). (*b*) Discuss how to start Richardson's method. (*c*) Discuss the stability of Richardson's method. (*d*) Test Richardson's method on the initial-boundary value problem in Exercise 5.2.2. Does your numerical solution conform with your expectations from part (*b*)?

5.3 Numerical Algorithms for First-Order PDE

5.3.1. Solve the initial value problem $u_t = 3u_x$, $u(0,x) = 1/(1+x^2)$, on the interval $[-10, 10]$ using an upwind scheme with space step size $\Delta x = .1$. Decide on an appropriate time step size, and graph your solution at times $t = .5, 1, 1.5$. Discuss what you observe.

5.3.2. Solve Exercise 5.3.1 for the nonuniform transport equations

(*a*) $u_t + 4(1+x^2)^{-1} u_x = 0$, (*b*) $u_t = \left(3 - 2e^{-x^2/4}\right) u_x$,

(*c*) $u_t + 7x\,(1+x^2)^{-1} u_x = 0$, (*d*) $u_t + \left(2\tan^{-1} \tfrac{1}{2}x\right) u_x = 0$.

5.3.3. Consider the initial value problem

$$u_t + \frac{3x}{x^2+1}\, u_x = 0, \qquad u(0,x) = \left(1 - \tfrac{1}{2}x^2\right)e^{-x^2/3}.$$

On the interval $[-5,5]$, using space step size $\Delta x = .1$ and time step size $\Delta t = .025$, apply (a) the forward scheme (5.38) (suitably modified for variable wave speed), (b) the backward scheme (5.44) (suitably modified for variable wave speed), and (c) the upwind scheme (5.49). Graph the resulting numerical solutions at times $t = .5, 1, 1.5$, and discuss what you observe in each case. Which of the schemes are stable?

5.3.4. Use the centered difference scheme (5.46) to solve the initial value problem in Exercise 5.3.1. Do you observe any instabilities in your numerical solution?

5.3.5. Use the Lax–Wendroff scheme (5.51) to solve the initial value problem in Exercise 5.3.1. Discuss the accuracy of your solution in comparison with the upwind scheme.

5.3.6. Can you explain why, in Figure 5.5, the numerical solution in the case $c = -1$ is significantly better than for $c = -.5$, or, indeed, for any other c in the stable range.

5.3.7. Nonlinear transport equations are often solved numerically by writing them in the form of a conservation law, and then applying the finite difference formulas directly to the conserved density and flux. (a) Devise an upwind scheme for numerically solving our favorite nonlinear transport equation, $u_t + \frac{1}{2}(u^2)_x = 0$.
(b) Test your scheme on the initial value problem $u(0,x) = e^{-x^2}$.

5.3.8. (a) Design a stable numerical solution scheme for the damped transport equation $u_t + \frac{3}{4}u_x + u = 0$. (b) Test your scheme on the initial value problem with $u(0,x) = e^{-x^2}$.

5.3.9. Analyze the stability of the numerical scheme (5.44) by applying (a) the CFL condition; (b) a von Neumann analysis. Are your conclusions the same?

5.3.10. For what choices of step size $\Delta t, \Delta x$ is the Lax–Wendroff scheme (5.51) stable?

5.4 Numerical Algorithms for the Wave equation

5.4.1. Suppose you are asked to numerically approximate the solution to the initial-boundary value problem

$$u_{tt} = 64\,u_{xx}, \quad u(t,0) = u(t,3) = 0, \quad u(0,x) = \begin{cases} 1 - 2|\,x-1\,|, & \frac{1}{2} \le x \le \frac{3}{2}, \\ 0, & \text{otherwise}, \end{cases} \quad u_t(0,x) = 0,$$

on the interval $0 \le x \le 3$, using (5.56) with space step size $\Delta x = .1$. (a) What range of time steps Δt are allowed? (b) Test your answer by implementing the numerical solution for one value of Δt in the allowable range and one value outside. Discuss what you observe in your numerical solutions. (c) In the stable range, compare your numerical solution with that obtained using the smaller step size $\Delta x = .01$ and a suitable time step Δt.

5.4.2. Solve Exercise 5.4.1 for the boundary value problem

$$u_{tt} = 64 u_{xx}, \quad u(t,0) = 0 = u(t,3), \quad u(0,x) = 0, \quad u_t(0,x) = \begin{cases} 1 - 2\,|\,x - 1\,|, & \frac{1}{2} \le x \le \frac{3}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

5.4.3. Solve the following initial-boundary value problem

$$u_{tt} = 9 u_{xx}, \quad u(t,0) = u(t,1) = 0, \quad u(0,x) = \tfrac{1}{2} + \left| x - \tfrac{1}{4} \right| - \left| 2x - \tfrac{3}{4} \right|, \quad u_t(0,x) = 0,$$

on the interval $0 \le x \le 1$, using the numerical scheme (5.56) with space step sizes $\Delta x = .1, .01$ and $.001$ and suitably chosen time steps. Discuss which features of the solution can be observed in your numerical approximations.

5.4.4. (a) Use a numerical integrator with space step size $\Delta x = .05$ to solve the periodically forced boundary value problem

$$u_{tt} = u_{xx}, \qquad u(0,x) = u_t(0,x) = 0, \qquad u(t,0) = \sin t, \qquad u(t,1) = 0.$$

Is your solution periodic? (b) Repeat the computation using the alternative boundary condition $u(t,0) = \sin \pi t$. Discuss any observed differences between the two problems.

5.4.5. (a) Design an explicit numerical scheme for solving the initial-boundary value problem

$$u_{tt} = c^2 u_{xx} + F(t,x), \quad u(t,0) = u(t,1) = 0, \quad u(0,x) = f(x), \quad u_t(0,x) = g(x), \quad 0 \le x \le 1,$$

for the wave equation with an external *forcing term* $F(t,x)$. Clearly state any stability conditions that need to be imposed on the time and space step sizes.

(b) Test your scheme on the particular case $c = \frac{1}{4}$, $F(t,x) = 3\,\mathrm{sign}\left(x - \frac{1}{2}\right) \sin \pi t$, $f(x) \equiv g(x) \equiv 0$, using space step sizes $\Delta x = .05$ and $.01$, and suitably chosen time steps.

5.4.6. Let $\beta > 0$. (a) Design a finite difference scheme for approximating the solution to the initial-boundary value problem

$$u_{tt} + \beta u_t = c^2 u_{xx}, \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x),$$

for the damped wave equation on the interval $0 \le x \le 1$. (b) Discuss the stability of your scheme. What choice of step sizes will ensure stability? (c) Test your scheme with $c = 1$, $\beta = 1$, using the initial data $f(x) = e^{-(x-.7)^2}$, $g(x) = 0$.

5.5 Finite Difference Algorithms for the Laplace and Poisson equations

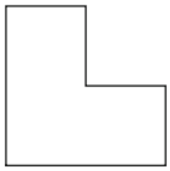
5.5.1. Solve the Dirichlet problem $\Delta u = 0$, $u(x,0) = \sin^3 x$, $u(x,\pi) = 0$, $u(0,y) = 0$, $u(\pi,y) = 0$, numerically using a finite difference scheme. Compare your approximation with the solution you obtained in Exercise 4.3.10(a).

5.5.2. Solve the Dirichlet problem $\Delta u = 0$, $u(x,0) = x$, $u(x,1) = 1 - x$, $u(0,y) = y$, $u(1,y) = 1 - y$, numerically via finite differences. Compare your approximation with the solution you obtained in Exercise 4.3.12(d).

5.5.3. Consider the Dirichlet boundary value problem $\Delta u = 0 \quad u(x,0) = \sin x,\ u(x,\pi) = 0,$ $u(0,y) = 0,\ u(\pi,y) = 0$, on the square $\{0 < x, y < \pi\}$. (a) Find the exact solution. (*b*) Set up and solve the finite difference equations based on a square mesh with $m = n = 2$ squares on each side of the full square. How close is this value to the exact solution at the center of the square: $u\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$? (*c*) Repeat part (*b*) for $m = n = 4$ squares per side. Is the value of your approximation at the center of the unit square closer to the true solution? (*d*) Use a computer to find a finite difference approximation to $u\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$ using $m = n = 8$ and 16 squares per side. Is your approximation converging to the exact solution as the mesh becomes finer and finer? Is the convergence rate consistent with the order of the finite difference approximation?

5.5.4. (a) Use finite differences to approximate a solution to the Helmholtz boundary value problem $\Delta u = u,\ u(x,0) = u(x,1) = u(0,y) = 0,\ u(1,y) = 1,$ on the unit square $0 < x, y < 1$. (*b*) Use separation of variables to construct a series solution. Do your analytic and numerical solutions match? Explain any discrepancies.

5.5.5. A drum is in the shape of an L, as in the accompanying figure, whose short sides all have length 1. (a) Use a finite difference scheme with mesh spacing $\Delta x = \Delta y = .1$ to find and graph the equilibrium configuration when the drum is subject to a unit upwards force while all its sides are fixed to the (x, y)–plane. What is the maximal deflection, and at which point(s) does it occur? (*b*) Check the accuracy of your answer in part (a) by reducing the step size by half: $\Delta x = \Delta y = .05$.



5.5.6. A metal plate has the shape of a 3 cm square with a 1 cm square hole cut out of the middle. The plate is heated by making the inner edge have temperature 100° while keeping the outer edge at 0°. (*a*) Find the (approximate) equilibrium temperature using finite differences with a mesh width of $\Delta x = \Delta y = .5$ cm. Plot your approximate solution using a three-dimensional graphics program. (*b*) Let C denote the square contour lying midway between the inner and outer square boundaries of the plate. Using your finite difference approximation, determine at what point(s) on C the temperature is (*i*) minimized; (*ii*) maximimized; (*iii*) equal to the average of the two boundary temperatures. (*c*) Repeat part (*a*) using a smaller mesh width of $\Delta x = \Delta y = .2$. How much does this affect your answers in part (*b*)?

5.5.7. Answer Exercise 5.5.6 when the plate is additionally subjected to a constant heat source

$$f(x,y) = 600x + 800y - 2400.$$

5.5.8. (*a*) Explain how to adapt the finite difference method to a mixed boundary value problem on a rectangle with inhomogeneous Neumann conditions. *Hint*: Use a one-sided difference formula of the appropriate order to approximate the normal derivative at the boundary. (*b*) Apply your method to the problem

$$\Delta u = 0, \qquad u(x,0) = 0, \qquad u(x,1) = 0, \qquad \frac{\partial u}{\partial x}(0,y) = y(1-y), \qquad u(1,y) = 0,$$

using mesh sizes $\Delta x = \Delta y = .1, .01$, and $.001$. Compare your answers. (*c*) Solve the boundary value problem via separation of variables, and compare the value of the solution and the numerical approximations at the center of the square.