Peter J. Olver

Ch4 Separation of Variables

4.1 The Diffusion and Heat Equations

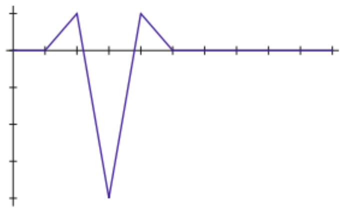
4.1.1. Suppose the ends of a bar of length 1 and thermal diffusivity $\gamma = 1$ are held fixed at respective temperatures 0° and 10°. (*a*) Determine the equilibrium temperature profile. (*b*) Determine the rate at which the equilibrium temperature profile is approached. (*c*) What does the temperature profile look like as it nears equilibrium?

4.1.2. A uniform insulated bar 1 meter long is stored at room temperature of 20° Celsius. An experimenter places one end of the bar in boiling water and the other end in ice water.
(*a*) Set up an initial-boundary value problem that models the temperature in the bar.
(*b*) Find the equilibrium temperature distribution.
(*c*) Discuss how your answer depends on the material properties of the bar.

4.1.3. Consider the initial-boundary value problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{array}{ll} u(t,0) = 0 = u(t,10), & t > 0, \\ u(0,x) = f(x), & 0 < x < 10, \end{array}$$

for the heat equation where the initial data has the following form:



$$f(x) = \begin{cases} x - 1, & 1 \le x \le 2, \\ 11 - 5x, & 2 \le x \le 3, \\ 5x - 19, & 3 \le x \le 4, \\ 5 - x, & 4 \le x \le 5, \\ 0, & \text{otherwise.} \end{cases}$$

Discuss what happens to the solution as t increases. You do *not* need to write down an explicit formula, but for full credit you must explain (sketches can help) at least three or four interesting things that happen to the solution as time progresses.

4.1.4. Find a series solution to the initial-boundary value problem for the heat equation $u_t = u_{xx}$ for $0 < x < 1$ when one the end of the bar is held at 0° and the other is insulated. Discuss the asymptotic behavior of the solution as $t \to \infty$.

4.1.5. Answer Exercise 4.1.4 when both ends of the bar are insulated.

4.1.6. A metal bar, of length $\ell = 1$ meter and thermal diffusivity $\gamma = 2$, is taken out of a 100° oven and then fully insulated except for *one* end, which is fixed to a large ice cube at 0°. (*a*) Write down an initial-boundary value problem that describes the temperature $u(t,x)$ of the bar at all subsequent times. (*b*) Write a series formula for the temperature distribution $u(t,x)$ at time $t > 0$. (*c*) What is the equilibrium temperature distribution in the bar, i.e., for $t \gg 0$? How fast does the solution go to equilibrium? (*d*) Just before the temperature distribution reaches equilibrium, what does it look like? Sketch a picture and discuss.

4.1.7. A metal bar of length $\ell = 1$ and thermal diffusivity $\gamma = 1$ is fully insulated, including its ends. Suppose the initial temperature distribution is $u(0,x) = \begin{cases} x, & 0 \le x \le \frac{1}{2}, \\ 1-x, & \frac{1}{2} \le x \le 1. \end{cases}$
(a) Use Fourier series to write down the temperature distribution at time $t > 0$.

(b) What is the equilibrium temperature distribution in the bar, i.e., for $t \gg 0$?
(c) How fast does the solution go to equilibrium? (d) Just before the temperature distribution reaches equilibrium, what does it look like? Sketch a picture and discuss.

4.1.8. (a) Find the series solution to the heat equation $u_t = u_{xx}$ on $-2 < x < 2$, $t > 0$, when subject to the Dirichlet boundary conditions $u(t,-2) = u(t,2) = 0$ and the initial condition $u(0,x) = \begin{cases} x, & |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$ (b) Sketch a graph of the solution at some representative times. (c) At what rate does the temperature approach thermal equilibrium?

4.1.9. Solve the heat equation when the right-hand end of a bar of unit length is held at a fixed constant temperature α while the left-hand end is insulated. Discuss the asymptotic behavior of the solution.

4.1.10. For each of the following initial temperature distributions, (i) write out the Fourier series solution to the heated ring (4.30–32), and (ii) find the resulting equilibrium temperature as $t \to \infty$: (a) $\cos x$, (b) $\sin^3 x$, (c) $|x|$, (d) $\begin{cases} 1, & -\pi < x < 0, \\ 0, & 0 < x < \pi. \end{cases}$

4.1.11. Suppose that the temperature $u(t,x)$ of a homogeneous bar satisfies the heat equation. Show that the associated heat flux $w(t,x)$ is also a solution to the same heat equation.

4.1.12. Show that the time derivative $v = u_t$ of any solution to the heat equation is also a solution. If $u(t,x)$ satisfies the initial condition $u(0,x) = f(x)$, what initial condition does $v(t,x)$ inherit?

4.1.13. Explain why the thermal energy $E(t) = \int_0^\ell u(t,x)\,dx$ is not constant for the Dirichlet initial-boundary value problem for the heat equation on the interval $[0,\ell]$.

4.1.14. (a) Show that the thermal energy $E(t) = \int_0^\ell u(t,x)\,dx$ is constant for the Neumann boundary value problem on the interval $[0,\ell]$. (b) Use part (a) to prove that the constant equilibrium solution for the homogeneous Neumann boundary value problem is equal to the mean initial temperature $u(0,x)$.

4.1.15. Let $u(t,x)$ be any nonconstant solution to the periodic heat equation (4.30–31). Prove that the squared L^2 norm of the solution, $N(t) = \int_{-\pi}^{\pi} u(t,x)^2\,dx$, is a strictly decreasing function of t. *Remark*: Interestingly, comparing this result with formula (4.38), we find that, for the periodic boundary value problem, the integral of u is constant, but the integral of u^2 is strictly decreasing. How is this possible?

4.1.16. The *cable equation* $v_t = \gamma v_{xx} - \alpha v$, with $\gamma, \alpha > 0$, also known as the *lossy heat equation*,was derived by the nineteenth-century Scottish physicist William Thomson to model propagation of signals in a transatlantic cable. Later, in honor of his work on thermodynamics, including determining the value of absolute zero temperature, he was named Lord Kelvin by Queen Victoria. The cable equation was later used to model the electrical activity of neurons. (a) Show that the general solution to the cable equation is given by $v(t,x) = e^{-\alpha t}\,u(t,x)$, where $u(t,x)$ solves the heat equation $u_t = \gamma u_{xx}$.
(b) Find a Fourier series solution to the Dirichlet initial-boundary value problem

$$v_t = \gamma\, v_{xx} - \alpha\, v, \qquad v(0,x) = f(x), \qquad v(t,0) = 0 = v(t,1), \qquad 0 \le x \le 1, \qquad t > 0.$$

Does your solution approach an equilibrium value? If so, how fast?
(c) Answer part (b) for the Neumann problem

$$v_t = \gamma\, v_{xx} - \alpha\, v, \qquad v(0,x) = f(x), \qquad v_x(t,0) = 0 = v_x(t,1), \qquad 0 \le x \le 1, \qquad t > 0.$$

4.1.17. The *convection-diffusion equation* $u_t + c u_x = \gamma u_{xx}$ is a simple model for the diffusion of a pollutant in a fluid flow moving with constant speed c. Show that $v(t,x) = u(t, x + ct)$ solves the heat equation. What is the physical interpretation of this change of variables?

4.1.18. Combine Exercises 4.1.16–17 to solve the *lossy convection-diffusion equation*

$$u_t = \gamma u_{xx} + c u_x - \alpha u.$$

4.1.19. Let $\gamma > 0$ and $\lambda \le 0$. (a) Find all solutions to the differential equation $\gamma v'' + \lambda v = 0$. ($b$) Prove that the only solution that satisfies the boundary conditions $v(0) = 0$, $v(\ell) = 0$, is the zero solution $v(x) \equiv 0$.

4.1.20. Answer Exercise 4.1.19 when λ is a non-real complex number.

4.2 The Wave Equations

4.2.1. In music, an octave corresponds to doubling the frequency of the sound waves. On my piano, the middle C string has length .7 meter, while the string for the C an octave higher has length .6 meter. Assuming that they have the same density, how much tighter does the shorter string need to be tuned?

4.2.2. How much longer would a piano string have to be to make the same sound when it is pulled twice as tight?

4.2.3. Write down the solutions to the following initial-boundary value problems for the wave equation in the form of a Fourier series:

(a) $u_{tt} = u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = 1$, $u_t(0,x) = 0$;
(b) $u_{tt} = 2u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = 0$, $u_t(0,x) = 1$;
(c) $u_{tt} = 3u_{xx}$, $u(t,0) = u(t,\pi) = 0$, $u(0,x) = \sin^3 x$, $u_t(0,x) = 0$;
(d) $u_{tt} = 4u_{xx}$, $u(t,0) = u(t,1) = 0$, $u(0,x) = x$, $u_t(0,x) = -x$;
(e) $u_{tt} = u_{xx}$, $u(t,0) = u_x(t,1) = 0$, $u(0,x) = 1$, $u_t(0,x) = 0$;
(f) $u_{tt} = 2u_{xx}$, $u_x(t,0) = u_x(t,2\pi) = 0$, $u(0,x) = -1$, $u_t(0,x) = 1$;
(g) $u_{tt} = u_{xx}$, $u_x(t,0) = u_x(t,1) = 0$, $u(0,x) = x(1-x)$, $u_t(0,x) = 0$.

4.2.4. Find all separable solutions to the wave equation $u_{tt} = u_{xx}$ on the interval $0 \le x \le \pi$ subject to (a) mixed boundary conditions $u(t,0) = 0$, $u_x(t,\pi) = 0$;
(b) Neumann boundary conditions $u_x(t,0) = 0$, $u_x(t,\pi) = 0$.

4.2.5. (a) Under what conditions is the solution to the Neumann boundary value problem (4.75) a periodic function of t? What is the period? (b) Establish explicit periodicity formulas of the form (4.74). (c) Under what conditions is the velocity $\partial u/\partial t$ periodic in t?

4.2.6. (a) Formulate the periodic initial-boundary value problem for the wave equation on the interval $-\pi \le x \le \pi$, modeling the vibrations of a circular ring. (b) Write out a formula for the solution to your problem in the form of a Fourier series. (c) Is the solution a periodic function of t? If so, what is the period? (d) Suppose the initial displacement coincides with that in Figure 4.6, while the initial velocity is zero. Describe what happens to the solution as time evolves.

4.2.7. Show that the time derivative, $v = \partial u/\partial t$, of any solution to the wave equation is also a solution. If you know the initial conditions of u, what initial conditions does v satisfy?

4.2.8. Find all the separable real solutions to the wave equation subject to a restoring force: $u_{tt} = u_{xx} - u$. Discuss their long-term behavior.

4.2.9. Let $a, c > 0$ be positive constants. The *telegrapher's equation* $u_{tt} + a\,u_t = c^2\,u_{xx}$ represents a damped version of the wave equation. Consider the Dirichlet boundary value problem $u(t,0) = u(t,1) = 0$, on the interval $0 \le x \le 1$, with initial conditions $u(0,x) = f(x)$, $u_t(0,x) = 0$. (a) Find all separable solutions to the telegrapher's equation that satisfy the boundary conditions. (b) Write down a series solution for the initial boundary value problem. (c) Discuss the long term behavior of your solution. (d) State a criterion that distinguishes overdamped from underdamped versions of the equation.

4.2.10. The fourth-order partial differential equation $u_{tt} = -u_{xxxx}$ is a simple model for a vibrating elastic beam. (a) Find all separable real solutions to the beam equation. (b) Show that any (complex) solution to the Schrödinger equation $i\,u_t = u_{xx}$ solves the beam equation.

4.2.11. The initial-boundary value problem

$$u_{tt} = -u_{xxxx}, \qquad \begin{matrix} u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0, & 0 < x < 1, \\ u(0,x) = f(x), \quad u_t(0,x) = 0, & t > 0, \end{matrix}$$

models the vibrations of an elastic beam of unit length with simply supported ends, subject to a nonzero initial displacement $f(x)$ and zero initial velocity. (a) What are the vibrational frequencies for the beam? (b) Write down the solution to the initial-boundary value problem as a Fourier series. (c) Does the beam vibrate periodically (i) for all initial conditions? (ii) for some initial conditions? (iii) for no initial conditions?

4.2.12. *Multiple choice*: The initial-boundary value problem

$$u_{tt} = u_{xxxx}, \qquad \begin{matrix} u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0, & 0 < x < 1, \\ u(0,x) = f(x), \quad u_t(0,x) = g(x), & t > 0, \end{matrix}$$

is well-posed for (a) $t > 0$; (b) $t < 0$; (c) all t; (d) no t. Explain your answer.

The d'Alembert Formula for Bounded Intervals

4.2.13. (a) Solve the initial-boundary value problem from Example 4.3 using the d'Alembert method.
(b) Verify that your solution coincides with the Fourier series solution derived above.
(c) Justify our earlier observation that, at each time t, the solution $u(t,x)$ is a piecewise affine function of x.

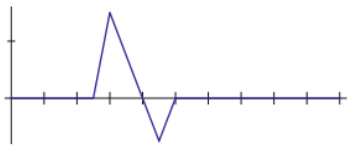
4.2.14. Sketch the solution of the wave equation $u_{tt} = u_{xx}$ and describe its behavior when the initial displacement is the box function $u(0,x) = \begin{cases} 1, & 1 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$ while the initial velocity is 0 in each of the following scenarios: (a) on the entire line $-\infty < x < \infty$; (b) on the half-line $0 \le x < \infty$, with homogeneous Dirichlet boundary condition at the end; (c) on the half-line $0 \le x < \infty$, with homogeneous Neumann boundary condition at the end; (d) on the bounded interval $0 \le x \le 5$ with homogeneous Dirichlet boundary conditions; (e) on the bounded interval $0 \le x \le 5$ with homogeneous Neumann boundary conditions.

4.2.15. Answer Exercise 4.2.14 when the initial velocity is the box function, while the initial displacement is zero.

4.2.16. Consider the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \begin{matrix} u(t,0) = 0 = u(t,10), & t > 0, \\ u(0,x) = f(x), \quad u_t(0,x) = 0, & 0 < x < 10, \end{matrix}$$

for the wave equation, where the initial data has the following form:



$$f(x) = \begin{cases} 3x - 7.5, & 2.5 \le x \le 3, \\ 6 - 1.5x, & 3 \le x \le 4.5, \\ 1.5x - 7.5, & 4.5 \le x \le 5, \\ 0, & \text{otherwise}. \end{cases}$$

Discuss what happens to the solution. You do *not* need to write down an explicit formula for the solution, but for full credit you must explain (sketches can help) at least three or four interesting things that happen to the solution as time progresses.

4.2.17. Repeat Exercise 4.2.16 for the Neumann boundary conditions.

4.2.18. Suppose the initial displacement of a string of length ℓ looks like the graph to the right. Assuming that the ends of the string are held fixed, graph the string's profile at times $t = \ell/c$ and $2\ell/c$.

4.2.19. Consider the wave equation $u_{tt} = u_{xx}$ on the interval $0 \le x \le 1$, with homogeneous Dirichlet boundary conditions at both ends. (a) Use the d'Alembert formula to explicitly solve the initial value problem $u(0,x) = x - x^2$, $u_t(0,x) = 0$. (*b*) Graph the solution profile at some representative times, and discuss what you observe. (*c*) Find the Fourier series at each t of your solution and compare the two. (*d*) How many terms do you need to sum to obtain a reasonable approximation to the exact solution?

4.2.20. Solve Exercise 4.2.19 for the initial conditions $u(0,x) = 0$, $u_t(0,x) = x^2 - x$.

4.2.21. Solve (*i*) Exercise 4.2.19, (*ii*) Exercise 4.2.20, when the solution is subject to homogeneous Neumann boundary conditions.

4.2.22. Under what conditions is the solution to the Neumann boundary value problem for the wave equation on a bounded interval $[0, \ell]$ periodic in time? What is the period?

4.2.23. Discuss and sketch the behavior of the solution to the Neumann boundary value problem $u_{tt} = 4\,u_{xx}$, $0 < x < 1$, $u_x(t,0) = 0 = u_x(t,1)$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$, for

(*a*) a localized initial displacement: $f(x) = \begin{cases} 1, & .2 < x < .3, \\ 0, & \text{otherwise.} \end{cases}$ $\quad g(x) = 0;$

(*b*) a localized initial velocity: $f(x) = 0$, $g(x) = \begin{cases} 1, & .2 < x < .3, \\ 0, & \text{otherwise.} \end{cases}$.

4.2.24. (a) Explain how to solve the Neumann initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(t,0) = 0 = \frac{\partial u}{\partial x}(t,1), \qquad u(0,x) = f(x), \qquad \frac{\partial u}{\partial t}(0,x) = g(x),$$

on the interval $0 \le x \le 1$.

(*b*) Let $f(x) = \begin{cases} x - \frac{1}{4}, & \frac{1}{4} \le x \le \frac{1}{2}, \\ \frac{3}{4} - x, & \frac{1}{2} \le x \le \frac{3}{4}, \\ 0, & \text{otherwise,} \end{cases}$ and $g(x) = 0$. Sketch the graph of the solution at a few representative times, and discuss what is happening. Is the solution periodic in time? If so, what is the period?

(*c*) Do the same when $f(x) = 0$ and $g(x) = x$.

4.2.25. (a) Write down a formula for the solution $u(t,x)$ to the initial-boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - 4\,\frac{\partial^2 u}{\partial x^2} = 0, \quad u(0,x) = \sin x, \quad \frac{\partial u}{\partial t}(0,x) = \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,\pi) = 0, \quad 0 < x < \pi, \quad t > 0.$$

(b) Find $u\left(\frac{\pi}{2},\frac{\pi}{2}\right)$. (c) Prove that $h(t) = u\left(t,\frac{\pi}{2}\right)$ is a periodic function of t and find its period. (d) Does $\dfrac{\partial u}{\partial x}$ have any discontinuities? If so, discuss their behavior.

4.2.26. Answer Exercise 4.2.25 for the mixed boundary conditions $u(t,0) = 0 = u_x(t,\pi)$.

4.2.27. (a) Explain how to use d'Alembert's formula (4.77) to solve the periodic initial-boundary value problem for the wave equation given in Exercise 4.2.6.
(b) Do the d'Alembert and Fourier series formulae represent the same solution? If so, can you justify it? If not, explain why they are different.

4.2.28. Show that the solution $u(t,x)$ to the wave equation on an interval $[0,\ell]$, subject to periodic boundary conditions $u(t,0) = u(t,\ell)$, $u_x(t,0) = u_x(t,\ell)$, is a periodic function of t if and only if there is no net initial velocity: $\int_0^\ell g(x)\,dx = 0$.

4.2.29. (a) Explain how to solve the wave equation on a half-line $x > 0$ when subject to Dirichlet boundary conditions $u(t,0) = 0$. (b) Assuming $c = 1$, find the solution satisfying $u(0,x) = (x-2)\,e^{-5\,(x-2.2)^2}$, $u_t(0,x) = 0$. (c) Sketch a picture of your solution at some representative times, and discuss what is happening.

4.2.30. Solve Exercise 4.2.29 for homogeneous Neumann boundary conditions at $x = 0$.

4.2.31. (a) Given that $f(x)$ is odd and 2ℓ–periodic, explain why $f(0) = 0 = f(\ell)$.
(b) Given that $f(x)$ is even and 2ℓ–periodic, explain why $f'(0) = 0 = f'(\ell)$.

4.2.32. (a) Prove that if $f(-x) = -f(x)$, $f(x+2\ell) = f(x)$, for all x, then $u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$ satisfies the Dirichlet boundary conditions (4.79).
(b) Prove that if $g(-x) = -g(x)$, $g(x+2\ell) = g(x)$ for all x, then $u(t,x) = \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(z)\,dz$ also satisfies the Dirichlet boundary conditions.

4.2.33. If both $u(0,x) = f(x)$ and $u_t(0,x) = g(x)$ are even functions, show that the solution $u(t,x)$ of the wave equation is even in x for all t.

4.2.34. (a) Prove that the solution $u(t,x)$ to the wave equation for $x \in \mathbb{R}$ is an even function of t if and only if its initial velocity, at $t = 0$, is zero.
(b) Under what conditions is $u(t,x)$ an odd function of t?

4.2.35. Let $u(t,x)$ be a classical solution to the wave equation $u_{tt} = c^2 u_{xx}$ on the interval $0 < x < \ell$, satisfying homogeneous Dirichlet boundary conditions. The *total energy* of u at time t is

$$E(t) = \int_0^\ell \frac{1}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right] dx. \tag{4.81}$$

Establish the *Law of Conservation of Energy* by showing that $E(t) = E(0)$ is a constant function.

4.2.36. (a) Use Exercise 4.2.35 to prove that the only C^2 solution to the initial-boundary value problem $v_{tt} = c^2 v_{xx}$, $v(t,0) = v(t,\ell) = 0$, $v(0,x) = 0$, $v_t(0,x) = 0$, is the trivial solution $v(t,x) \equiv 0$. (b) Establish the following *Uniqueness Theorem* for the wave equation: given $f(x), g(x) \in C^2$, there is at most one C^2 solution $u(t,x)$ to the initial-boundary value problem $u_{tt} = c^2 u_{xx}$, $u(t,0) = u(t,\ell) = 0$, $u(0,x) = f(x)$, $u_t(0,x) = g(x)$.

4.2.37. Referring back to Exercises 4.2.35 and 4.2.36: (a) Does conservation of energy hold for solutions to the homogeneous Neumann initial-boundary value problem?
(b) Can you establish a uniqueness theorem for the Neumann problem?

4.2.38. Explain how to solve the Dirichlet initial-boundary value problem

$$u_{tt} = c^2 u_{xx} + F(t,x), \qquad u(0,x) = f(x), \qquad u_t(0,x) = g(x), \qquad u(t,0) = u(t,\ell) = 0,$$

for the wave equation subject to an external forcing on the interval $[0,\ell]$.

4.3 The Planar Laplace and Poisson Equations

4.3.1. (a) Solve the boundary value problem $\Delta u = 1$ for $x^2 + y^2 < 1$ and $u(x,y) = 0$ for $x^2 + y^2 = 1$ directly. *Hint*: The solution is a simple polynomial.
(b) Graph your solution, interpreting it as the equilibrium displacement of a circular drum under a constant gravitational force.

4.3.2. Set up the boundary value problem corresponding to the equilibrium of a circular membrane subject to a constant downwards gravitational force, half of whose boundary is glued to a flat semicircular wire, while the other half is unattached.

4.3.3. Set up the boundary value problem corresponding to the thermal equilibrium of a rectangular plate that is insulated on two of its sides, has 0° at its top edge and 100° at the bottom edge. Where do you expect the maximum temperature to be located? What is its value? Can you find a formula for the temperature inside the plate? *Hint*: The solution is constant along horizontal lines.

4.3.4. Set up the boundary value problem corresponding to the thermal equilibrium of an insulated semi-circular plate with unit diameter, whose curved edge is kept at 0° and whose straight edge is at 50°.

4.3.5. Explain why the solution to the homogeneous Neumann boundary value problem for the Laplace equation is *not* unique.

4.3.6. Write down the Dirichlet boundary value problem for the Laplace equation on the unit square $0 \le x, y \le 1$ that is satisfied by $u(x,y) = 1 + xy$.

4.3.7. Write down the Neumann boundary value problem for the Poisson equation on the unit disk $x^2 + y^2 \le 1$ that is satisfied by $u(x,y) = x^3 + xy^2$.

4.3.8. Suppose $u(x,y)$ is a solution to the Laplace equation.
(a) Show that any translate $U(x,y) = u(x-a, y-b)$, where $a, b \in \mathbb{R}$, is also a solution.
(b) Show that the rotated function $U(x,y) = u(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$, where $-\pi < \theta \le \pi$, is also a solution.

4.3.9. (a) Show that if $u(x,y)$ solves the Laplace equation, then so does the rescaled function $U(x,y) = c\,u(\alpha x, \alpha y)$ for any constants c, α.
(b) Discuss the effect of scaling on the Dirichlet boundary value problem.
(c) What happens if we use different scaling factors in x and y?

Separation of Variables

4.3.10. Solve the following boundary value problems for Laplace's equation on the square
$$\Omega = \{0 \le x \le \pi, \quad 0 \le y \le \pi\}.$$
(a) $u(x,0) = \sin^3 x, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u(\pi,y) = 0.$
(b) $u(x,0) = 0, \quad u(x,\pi) = 0, \quad u(0,y) = \sin y, \quad u(\pi,y) = 0.$
(c) $u(x,0) = 0, \quad u(x,\pi) = 1, \quad u(0,y) = 0, \quad u(\pi,y) = 0.$
(d) $u(x,0) = 0, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u(\pi,y) = y(\pi - y).$

4.3.11. (a) Explain how to use linear superposition to solve the boundary value problem
$$\Delta u = 0, \qquad u(x,0) = f(x), \qquad u(x,b) = g(x), \qquad u(0,y) = h(y), \qquad u(a,y) = k(y),$$
on the rectangle $R = \{0 < x < a,\ 0 < y < b\}$, by splitting it into four separate boundary value problems for which each of the solutions vanishes on three sides of the rectangle.
(b) Write down a series formula for the resulting solution.

4.3.12. Solve the following Dirichlet problems for Laplace's equation on the unit square $S = \{0 < x, y < 1\}$. *Hint*: Use superposition as in Exercise 4.3.11.
(a) $u(x,0) = \sin \pi x, \quad u(x,1) = 0, \quad u(0,y) = \sin \pi y, \quad u(1,y) = 0;$
(b) $u(x,0) = 1, \quad u(x,1) = 0, \quad u(0,y) = 1, \quad u(1,y) = 0;$
(c) $u(x,0) = 1, \quad u(x,1) = 1, \quad u(0,y) = 0, \quad u(1,y) = 0;$
(d) $u(x,0) = x, \quad u(x,1) = 1 - x, \quad u(0,y) = y, \quad u(1,y) = 1 - y.$

4.3.13. Solve the following mixed boundary value problems for Laplace's equation $\Delta u = 0$ on the square $S = \{0 < x, y < \pi\}$.
(a) $u(x,0) = \sin \frac{1}{2}x, \quad u_y(x,\pi) = 0, \quad u(0,y) = 0, \quad u_x(\pi,y) = 0;$
(b) $u(x,0) = \sin \frac{1}{2}x, \quad u_y(x,\pi) = 0, \quad u_x(0,y) = 0, \quad u_x(\pi,y) = 0;$
(c) $u(x,0) = x, \quad u(x,\pi) = 0, \quad u_x(0,y) = 0, \quad u_x(\pi,y) = 0;$
(d) $u(x,0) = x, \quad u(x,\pi) = 0, \quad u(0,y) = 0, \quad u_x(\pi,y) = 0.$

4.3.14. Find the solution to the boundary value problem

$$\Delta u = 0, \qquad \begin{array}{c} u_y(x,0) = u_y(x,2) = 0, \\ u(0,y) = 2\cos \pi y - 1, \quad u(1,y) = 0, \end{array} \qquad \begin{array}{c} 0 < x < 1, \\ 0 < y < 2. \end{array}$$

4.3.15. Find the solution to the boundary value problem

$$\Delta u = 0, \qquad \begin{array}{c} u(x,0) = 2\cos 7\pi x - 4, \quad u(x,1) = 5\cos 3\pi x, \\ u_x(0,y) = u_x(1,y) = 0, \end{array} \qquad 0 < x, y < 1.$$

4.3.16. Let $u(x,y)$ be the solution to the boundary value problem

$$\Delta u = 0,\ u(x,-1) = f(x),\ u(x,1) = 0,\ u(-1,y) = 0,\ u(1,y) = 0,\ -1 < x < 1,\ -1 < y < 1.$$

(a) *True or false*: If $f(-x) = -f(x)$ is odd, then $u(0,y) = 0$ for all $-1 \le y \le 1$.
(b) *True or false*: If $f(0) = 0$, then $u(0,y) = 0$ for all $-1 \le y \le 1$.
(c) Under what conditions on $f(x)$ is $u(x,0) = 0$ for all $-1 \le x \le 1$?

4.3.17. Use separation of variables to solve the following boundary value problem:

$$u_{xx} + 2u_y + u_{yy} = 0, \quad u(x,0) = 0, \quad u(x,1) = f(x), \quad u(0,y) = 0, \quad u(1,y) = 0.$$

4.3.18. Use separation of variables to solve the Helmholtz boundary value problem $\Delta u = u$, $u(x,0) = 0$, $u(x,1) = f(x)$, $u(0,y) = 0$, $u(1,y) = 0$, on the unit square $0 < x, y < 1$.

4.3.19. Provide the details for the derivation of (4.94).

4.3.20. Justify the statement that if $|\,b_n\,| \le M$ are uniformly bounded, then the coefficients given in (4.100) go to zero exponentially fast as $n \to \infty$ for any $0 < y \le b$.

4.3.21. Let $u(x,y)$ denote the solution to the boundary value problem (4.91–92). (a) Write down the Fourier sine series for $\partial u/\partial y$. (b) Prove that $\partial u/\partial y$ is an infinitely differentiable function of x. (c) Justify the same result for the functions $\partial^k u/\partial y^k$ for each $k \ge 0$. *Hint*: Don't forget that $u(x,y)$ solves the Laplace equation.

Polar Coordinates

4.3.22. Solve the following Euler differential equations by use of the power ansatz:
(a) $x^2 u'' + 5xu' - 5u = 0$, (b) $2x^2 u'' - xu' - 2u = 0$, (c) $x^2 u'' - u = 0$,
(d) $x^2 u'' + xu' - 3u = 0$, (e) $3x^2 u'' - 5xu' - 3u = 0$, (f) $\dfrac{d^2u}{dx^2} + \dfrac{2}{x}\dfrac{du}{dx} = 0$.

4.3.23. (*i*) Show that if $u(x)$ solves the *Euler differential equation*

$$a x^2 \frac{d^2u}{dx^2} + b x \frac{du}{dx} + c u = 0, \tag{4.133}$$

then $v(y) = u(e^y)$ solves a linear constant-coefficient differential equation.
(*ii*) Use this technique to solve the Euler differential equations in Exercise 4.3.22.

4.3.24. (a) Use the method in Exercise 4.3.23 to solve an Euler equation whose characteristic equation has a double root $r_1 = r_2 = r$. (*b*) Solve the specific equations

$$(i)\ x^2 u'' - x u' + u = 0, \qquad (ii)\ \frac{d^2u}{dx^2} + \frac{1}{x}\frac{du}{dx} = 0.$$

4.3.25. Solve the following boundary value problems:
(*a*) $\Delta u = 0,\ x^2 + y^2 < 1, \quad u = x^3,\ x^2 + y^2 = 1;$
(*b*) $\Delta u = 0,\ x^2 + y^2 < 2, \quad u = \log(x^2 + y^2),\ x^2 + y^2 = 1;$
(*c*) $\Delta u = 0,\ x^2 + y^2 < 4, \quad u = x^4,\ x^2 + y^2 = 4;$
(*d*) $\Delta u = 0,\ x^2 + y^2 < 1, \quad \dfrac{\partial u}{\partial \mathbf{n}} = x,\ x^2 + y^2 = 1.$

4.3.26. Let $u(x, y)$ be the solution to the boundary value problem $u_{xx} + u_{yy} = 0,\ x^2 + y^2 < 1,$ $u(x, y) = x^2,\ x^2 + y^2 = 1$. Find $u(0, 0)$.

4.3.27. (*a*) Find the equilibrium temperature on a disk of radius 1 when half the boundary is held at 1° and the other half is held at -1°. (*b*) Find the equilibrium temperature on a half-disk of radius 1 when the temperature is held to 1° on the curved edge and 0° on the straight edge. (*c*) Find the equilibrium temperature on a half disk of radius 1 when the temperature is held to 0° on the curved edge and 1° on the straight edge.

4.3.28. Find the solution to Laplace's equation $u_{xx} + u_{yy} = 0$ on the semi-disk $x^2 + y^2 < 1$, $y > 0$, that satisfies the boundary conditions $u(x, 0) = 0$ for $-1 < x < 1$ and $u(x, y) = y^3$ for $x^2 + y^2 = 1,\ y > 0$.

4.3.29. Find the equilibrium temperature on a half-disk of radius 1 when the temperature is held to 1° on the curved edge, while the straight edge is insulated.

4.3.30. Solve the Dirichlet boundary value problem for the Laplace equation on the pie wedge $W = \{0 < \theta < \frac{1}{4}\pi,\ 0 < r < 1\}$, when the nonzero boundary data $u(1, \theta) = h(\theta)$ appears only on the curved portion of its boundary.

4.3.31. Find a harmonic function $u(x, y)$ defined on the annulus $\frac{1}{2} < r < 1$ subject to the constant Dirichlet boundary conditions $u = a$ on $r = \frac{1}{2}$ and $u = b$ on $r = 1$.

4.3.32. Boiling water flows continually through a long circular metal pipe of inner radius 1 cm and outer radius 1.2 cm placed in an ice water bath. *True or false*: The temperature at the midpoint, at radius 1.1 cm, is 50°. If false, what is the temperature at this point?

4.3.33. Write out the series solution to the boundary value problem $u(1,\theta)=0,\ u(2,\theta)=h(\theta)$, for the Laplace equation on an annulus $1<r<2$. *Hint*: Use all of the separable solutions listed in (4.114).

4.3.34. Solve the following boundary value problems for the Laplace equation on the annulus $1<r<2$: (a) $u(1,\theta)=0,\ u(2,\theta)=1$, (b) $u(1,\theta)=0,\ u(2,\theta)=\cos\theta$, (c) $u(1,\theta)=\sin 2\theta,\ u(2,\theta)=\cos 2\theta$, (d) $u_r(1,\theta)=0,\ u(2,\theta)=1$, (e) $u_r(1,\theta)=0,\ u(2,\theta)=\sin 2\theta$, (f) $u_r(1,\theta)=0,\ u_r(2,\theta)=1$, (g) $u_r(1,\theta)=2,\ u_r(2,\theta)=1$.

4.3.35. Solve the following boundary value problems for the Laplace equation on the semi-annular domain $D=\{1<x^2+y^2<2,\ y>0\}$:
(a) $u(x,y)=0,\quad x^2+y^2=1,\quad u(x,y)=1,\quad x^2+y^2=2,\quad u(x,0)=0;$
(b) $u(x,y)=0,\quad x^2+y^2=1$ or $2,\quad u(x,0)=0,\quad x>0,\quad u(x,0)=1,\quad x<0.$

4.3.36. Solve the following boundary value problem:
$(x^2+y^2)(u_{xx}+u_{yy})+2xu_x+2yu_y=0,\ x^2+y^2<1,\ u(x,y)=1+3x,\ x^2+y^2=1.$

4.3.37. Justify the chain rule computation (4.104). Then justify formula (4.105) for the Laplacian in polar coordinates.

4.3.38. Suppose $\int_{-\pi}^{\pi} |\,h(\theta)\,|\,d\theta<\infty$. Prove that (4.115) converges uniformly to the solution to the boundary value problem (4.101) on any smaller disk $D_{r_\star}=\{r\le r_\star<1\}\subsetneq D_1$.

4.3.39. Prove directly that (4.124) satisfies the boundary conditions (4.122).

4.3.40. Justify the integration formula in (4.128).

4.3.41. Provide a complete proof that (4.129) is indeed the solution to the boundary value problem (4.127).

4.3.42. Complete the proof of Theorem 4.9 by showing that $u(x,y)=M^\star$ for all $(x,y)\in\Omega$. *Hint*: Join (x_0,y_0) to (x,y) by a curve $C\subset\Omega$ of finite length, and use the preceding part of the proof to inductively deduce the existence of a finite sequence of points $(x_i,y_i)\in C$, $i=0,\dots,n$, with $(x_n,y_n)=(x,y)$, and such that $u(x_i,y_i)=M^\star$.

4.3.43. Derive the analogue of the Poisson integral formula for the solution to the Neumann boundary value problem $\Delta u = 0,\ x^2 + y^2 < 1,\ \partial u/\partial \mathbf{n} = h,\ x^2 + y^2 = 1$, on the unit disk. Pay careful attention to the existence and uniqueness of solutions in your formulation.

4.3.44. Give an example of a solution to Poisson's equation on the unit disk that achieves its maximum at an interior point. Interpret your construction physically.

4.3.45. Let $p(x, y)$ be a polynomial (not necessarily harmonic). Suppose $u(x, y)$ is harmonic and equals $p(x, y)$ on the unit circle $x^2 + y^2 = 1$. Prove that $u(x, y)$ is a harmonic polynomial.

4.3.46. Write down an integral formula for the solution to the Dirichlet boundary value problem on a disk of radius $R > 0$, namely, $\Delta u = 0,\ x^2 + y^2 < R^2,\ u = h,\ x^2 + y^2 = R^2$.

4.3.47. State and prove a one-dimensional version of Theorem 4.8. Does the analogue of Theorem 4.9 hold?

4.3.48. A unit area square plate has 100° temperature on its top edge and 0° on its three other edges. *True or false*: The temperature at the center equals the average edge temperature.

4.3.49. Let $u(x, y)$ be a harmonic function on the unit disk with boundary values $h(\theta)$ when $r = 1$. Using the fact that (4.115) is the Taylor series for $u(x, y)$ at the origin: (a) Find integral formulas for its partial derivatives $u_x(0,0),\ u_y(0,0)$, involving the boundary values $h(\theta)$. (b) Generalize part (a) to the second-order derivatives $u_{xx}(0,0),\ u_{xy}(0,0),\ u_{yy}(0,0)$.

4.3.50. Prove that if $u(x, y)$ is a bounded harmonic function defined on all of $\mathbb{R}^2$, then u is constant. *Hint*: First generalize Exercise 4.3.49(a) to find the value of its gradient, $\nabla u(x_0, y_0)$, in terms of the values of u on a circle of radius a centered at (x_0, y_0). Then see what happens when the radius of the circle goes to ∞.

4.4 Classification

4.4.1. Plot the following conic sections and classify their type:
(a) $x^2 + 3y^2 = 1$, (b) $xy + x + y = 4$, (c) $x^2 - xy + y^2 = x - 2y$,
(d) $x^2 + 2xy + y^2 + y = 1$, (e) $x^2 - 2y^2 = 6x + 8y + 1$.

4.4.2. Determine the type of the following partial differential equations:
(a) $u_{tt} + 3u_{xx} = 0$, (b) $u_{tx} + u_t + u_x = u$, (c) $u_{tt} + u_t + u_x = 0$,
(d) $u_{tt} - u_{tx} + u_{xx} = u$, (e) $u_{tt} + 4u_{tx} + 4u_{xx} = u_t$, (f) $u_{tx} + u_{xx} = 0$.

4.4.3. Consider the partial differential equation $x\,u_{tt} + (t + x)\,u_{xx} = 0$. At what points of the plane is the equation elliptic? hyperbolic? parabolic? degenerate?

4.4.4. Answer Exercise 4.4.3 for the equations
(a) $x^2\,u_{xx} + x\,u_x + u_{yy} = 0$, (b) $\partial_x(x\,u_x) = \partial_y(y\,u_y)$, (c) $u_t = \partial_x[(x + t)u_x]$,
(d) $\nabla \cdot (c(x, y)\nabla u) = u$, where $c(x, y)$ is a given function.

4.4.5. Steady flow of air past an airplane is modeled by the partial differential equation $(m^2 - 1)u_{xx} + u_{yy} = 0$, in which x is the flight direction, y the transverse direction, and $m \geq 0$ is the *Mach number* — the ratio of the airplane's speed to the speed of sound. Show that the equation is hyperbolic for subsonic flight, but elliptic for supersonic flight.

4.4.6. Show that the second-order partial differential equation

$$-\frac{\partial}{\partial x}\left(p(x, y)\,\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x, y)\,\frac{\partial u}{\partial y}\right) + r(x, y)\,u = f(x, y)$$

is elliptic if and only if $p(x, y)$ and $q(x, y)$ are nonzero and have the same sign.

4.4.7. *True or false*: The type of a linear second-order partial differential equation is not affected by a change of independent variables: $\tau = \varphi(t, x)$, $\xi = \psi(t, x)$.

4.4.8. Let $v(t, x) = a(t, x)\,u(t, x) + b(t, x)$, where a, b are fixed functions with $a \neq 0$. Suppose u is a solution to a second-order linear partial differential equation. Prove that v also solves a linear partial differential equation of the same type.

4.4.9. *True or false*: The polar coordinate form (4.105) of the Laplace equation is elliptic.

4.4.10. Rewrite the Laplace equation $u_{xx} + u_{yy} = 0$ in terms of *parabolic coordinates* ξ, η, as defined by the equations $x = \xi^2 - \eta^2$, $y = 2\xi\,\eta$. Is the resulting equation elliptic?

4.4.11. Prove that the complex change of variables $x = x$, $t = \mathrm{i}\,y$, maps the Laplace equation $u_{xx} + u_{yy} = 0$ to the wave equation $u_{tt} = u_{xx}$. Explain why the type of a partial differential equation is *not* necessarily preserved under a complex change of variables.

4.4.12. Suppose, against all advice, we pose the elliptic Laplace equation as an initial value problem, namely

$$u_{tt} = -\,u_{xx} \qquad \text{for} \qquad 0 < x < 1, \qquad t > 0,$$
$$u(0, x) = f(x), \qquad u_t(0, x) = 0, \qquad 0 \leq x \leq 1, \qquad u(t, 0) = 0 = u(t, 1), \qquad t \geq 0.$$

(a) Prove that for any positive integer $n > 0$, the function $u_n(t, x) = \dfrac{\sin n\pi t\ \cosh n\pi x}{n}$ satisfies the initial value problem. Determine the initial condition $u_n(0, x) = f_n(x)$.
(b) Prove that, as $n \to \infty$, the initial condition $f_n(x) \to 0$ becomes vanishingly small, whereas, at any $t > 0$, the solution value $u_n\left(t, \frac{1}{2}\right) \to \infty$.
(c) Explain why this represents an ill-posed problem.

4.4.13. The *minimal surface equation* $(1+u_x^2)u_{xx}-2u_xu_yu_{xy}+(1+u_y^2)u_{yy}=0$ is (a) hyperbolic, (b) parabolic, (c) elliptic, (d) singular, (e) of variable type depending on the point in the domain, or (f) of variable type depending on the solution and the point in the domain.

Charateristics and the Cauchy Problem

4.4.14. Find and graph the real characteristic curves for each of the partial differential equations in Exercise 4.4.2.

4.4.15. Graph the characteristic curves for the Tricomi equation (4.137) in its hyperbolic region. What happens to the characteristics as one approaches the parabolic transition boundary?

4.4.16. *True or false*: The characteristic curves of the *Helmholtz equation* $u_{xx}+u_{yy}-u=0$ are circles.

4.4.17. (a) At what points of the plane is the partial differential equation $x\,u_{xx}+y\,u_{yy}=0$ elliptic? parabolic? hyperbolic? (b) How many characteristics are there through the point $(1,-1)$? (c) Find them explicitly.

4.4.18. Consider the partial differential equation $u_{xx}+y\,u_{xy}=y^2$.
(a) On which regions of the (x,y)–plane is the equation elliptic? parabolic? hyperbolic?
(b) Find the characteristics in the hyperbolic region.
(c) Find the general solution in the hyperbolic region. *Hint*: Use characteristic coordinates.

4.4.19. Find a partial differential equation whose characteristic curves are:
(a) the lines $x-y=a$, $x+2y=b$, where $a,b\in\mathbb{R}$ are arbitrary constants;
(b) the exponential curves $y=ce^x$ for $c\in\mathbb{R}$;
(c) the concentric circles $x^2+y^2=a$ for $a\geq 0$, and the rays $y=bx$.

4.4.20. Prove that any reparametrization of a characteristic curve for a given second-order linear partial differential equation is also a characteristic curve.

4.4.21. *True or false*: You can uniquely recover a second-order partial differential equation by knowing all its characteristic curves.

4.4.22. Prove that any invertible change of variables, as in Exercise 4.4.7, maps the characteristic curves of the original linear partial differential equation to the characteristic curves of the transformed equation. Thus, characteristic curves are intrinsic: they do not depend on the parametrization, nor on the coordinates used to represent the partial differential equation.