Introduction to Partial Differential Equations Peter J. Olver

Ch 3 Fourier Series

3.1 Eigensolutions of Linear Evolution Equations

3.1.1. For each of the following differential operators, $(i)$ prove linearity; $(ii)$ prove (3.5); $(iii)$ write down the corresponding linear evolution equation (3.2):
(a) $\dfrac{\partial}{\partial x}$, (b) $\dfrac{\partial}{\partial x}+1$, (c) $\dfrac{\partial^2}{\partial x^2}+3\,\dfrac{\partial}{\partial x}$, (d) $\dfrac{\partial}{\partial x}\,e^x\,\dfrac{\partial}{\partial x}$, (e) $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$.

3.1.2. Find all separable eigensolutions to the heat equation $u_t = u_{xx}$ on the interval $0 \le x \le \pi$ subject to (a) homogeneous Dirichlet boundary conditions $u(t,0)=0,\ u(t,\pi)=0$;
(b) mixed boundary conditions $u(t,0)=0,\ u_x(t,\pi)=0$;
(c) Neumann boundary conditions $u_x(t,0)=0,\ u_x(t,\pi)=0$.

3.1.3. Complete the table of eigensolutions to the heat equation, in the absence of boundary conditions, by allowing the eigenvalue $\lambda$ to be complex.

3.1.4. Find all separable eigensolutions to the following partial differential equations:
(a) $u_t = u_x$, (b) $u_t = u_x - u$, (c) $u_t = x\,u_x$.

3.1.5. (a) Find the real eigensolutions to the damped heat equation $u_t = u_{xx} - u$. (b) Which solutions satisfy the periodic boundary conditions $u(t,-\pi) = u(t,\pi),\ u_x(t,-\pi) = u_x(t,\pi)$?

3.1.6. Answer Exercise 3.1.5 for the diffusive transport equation $u_t + c\,u_x = u_{xx}$ modeling the combined diffusion and transport of a solute in a uniform flow with constant wave speed $c$.

3.1.7. (a) Find the real eigensolutions to the diffusion equation $u_t = (x^2\,u_x)_x$ modeling diffusion in an inhomogeneous medium on the half-line $x > 0$.
(b) Which solutions satisfy the Dirichlet boundary conditions $u(t,1) = u(t,2) = 0$?

Fourier Series

3.2.1. Find the Fourier series of the following functions: (a) $\operatorname{sign} x$, (b) $|\,x\,|$,
(c) $3x-1$, (d) $x^2$, (e) $\sin^3 x$, (f) $\sin x\cos x$, (g) $|\sin x\,|$, (h) $x\cos x$.

3.2.2. Find the Fourier series of the following functions:

(a) $\begin{cases} 1, & |\,x\,| < \frac12\pi, \\ 0, & \text{otherwise}, \end{cases}$ (b) $\begin{cases} 1, & \frac12\pi < |\,x\,| < \pi, \\ 0, & \text{otherwise}, \end{cases}$ (c) $\begin{cases} 1, & \frac12\pi < x < \pi, \\ 0, & \text{otherwise}, \end{cases}$

(d) $\begin{cases} x, & |\,x\,| < \frac12\pi, \\ 0, & \text{otherwise}, \end{cases}$ (e) $\begin{cases} \cos x, & |\,x\,| < \frac12\pi, \\ 0, & \text{otherwise}. \end{cases}$

3.2.3. Find the Fourier series of $\sin^2 x$ and $\cos^2 x$ without directly calculating the Fourier coefficients. *Hint*: Use some standard trigonometric identities.

3.2.4. Let $g(x) = \frac{1}{2} p_0 + \sum_{k=1}^{n} (p_k \cos k x + q_k \sin k x)$ be a trigonometric polynomial. Explain why its Fourier coefficients are $a_k = p_k$ and $b_k = q_k$ for $k \le n$, while $a_k = b_k = 0$ for $k > n$.

3.2.5. *True or false*: (a) The Fourier series for the function $2 f(x)$ is obtained by multiplying each term in the Fourier series for $f(x)$ by 2. (*b*) The Fourier series for the function $f(2x)$ is obtained by replacing $x$ by $2x$ in the Fourier series for $f(x)$. (*c*) The Fourier coefficients of $f(x) + g(x)$ can be found by adding the corresponding Fourier coefficients of $f(x)$ and $g(x)$. (*d*) The Fourier coefficients of $f(x)\, g(x)$ can be found by multiplying the corresponding Fourier coefficients of $f(x)$ and $g(x)$.

## Periodic Extensions

3.2.6. Graph the $2\pi$–periodic extension of each of the following functions. Which extensions are continuous? Differentiable? (*a*) $x^2$, (*b*) $(x^2 - \pi^2)^2$, (*c*) $e^x$, (*d*) $e^{-|x|}$, (*e*) $\sinh x$, (*f*) $1 + \cos^2 x$; (*g*) $\sin \frac{1}{2} \pi x$, (*h*) $\dfrac{1}{x}$, (*i*) $\dfrac{1}{1 + x^2}$.

3.2.7. Sketch a graph of the $2\pi$–periodic extension of each of the functions in Exercise 3.2.2.

3.2.8. Complete the proof of Lemma 3.4 by showing that $\widetilde{f}(x)$ is $2\pi$ periodic.

3.2.9. Suppose $f(x)$ is periodic with period $\ell$ and integrable. Prove that, for any $a$,

$$(a)\ \int_a^{a+\ell} f(x)\, dx = \int_0^{\ell} f(x)\, dx, \qquad (b)\ \int_0^{\ell} f(x + a)\, dx = \int_0^{\ell} f(x)\, dx.$$

3.2.10. Let $f(x)$ be a sufficiently nice $2\pi$–periodic function. (*a*) Prove that $f'(x)$ is $2\pi$–periodic. (*b*) Show that if $f(x)$ has *mean zero*, so $\int_{-\pi}^{\pi} f(x)\, dx = 0$, then $g(x) = \int_0^x f(y)\, dy$ is $2\pi$–periodic; (*c*) Does the result in part (*b*) rely on the fact that the lower limit in the integral for $g(x)$ is 0? (*d*) More generally, prove that if $f(x)$ has mean $m = \dfrac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$, then the function $g(x) = \int_0^x f(y)\, dy - m\, x$ is $2\pi$–periodic.

3.2.11. Given a function $f(x)$ defined for $0 \le x < \ell$, prove that there is a unique periodic function of period $\ell$ that agrees with $f$ on the interval $[0, \ell)$. If $\ell = 2\pi$, is this the same periodic extension as we constructed in the text? Explain your answer. Try the case $f(x) = x$ as an illustrative example.

3.2.12. Use the method in Exercise 3.2.11 to construct and graph the 1–periodic extensions of the following functions: (a) $x^2$, (b) $e^{-x}$, (c) $\cos \pi x$, (d) $\begin{cases} 1, & |x| < \frac{1}{2}\pi, \\ 0, & \text{otherwise.} \end{cases}$

3.2.13. (a) How many terms in Gregory's series (3.43) are required to compute the first two decimal digits of $\pi$? (b) The first 10 decimal digits? *Hint*: Use the fact that it is an alternating series. (c) For part (a), try summing up the required number of terms on your computer, and check whether you obtain an accurate result.

## Piecewise Continuous Functions

3.2.14. Find the discontinuities and the jump magnitudes for the following piecewise continuous functions:

(a) $2\sigma(x) + \sigma(x+1) - 3\sigma(x-1)$, (b) $\operatorname{sign}(x^2 - 2x)$, (c) $\sigma(x^2 - 2x)$, (d) $|x^2 - 2x|$, (e) $\sqrt{|x-2|}$, (f) $\sigma(\sin x)$, (g) $\operatorname{sign}(\sin x)$, (h) $|\sin x|$, (i) $e^{\sigma(x)}$, (j) $\sigma(e^x)$, (k) $e^{|x-2|}$.

3.2.15. Graph the following piecewise continuous functions. List all discontinuities and jump magnitudes.

(a) $\begin{cases} e^x, & 1 < |x| < 2, \\ 0, & \text{otherwise,} \end{cases}$ (b) $\begin{cases} \sin x, & 0 < x < \frac{1}{2}\pi, \\ 0, & \text{otherwise,} \end{cases}$ (c) $\begin{cases} \dfrac{\sin x}{x}, & 0 < |x| < 2\pi, \\ 1, & x = 0, \\ 0, & \text{otherwise,} \end{cases}$

(d) $\begin{cases} x & |x| \le 1, \\ x^2, & |x| > 1, \end{cases}$ (e) $\begin{cases} x, & -1 < x < 0, \\ \sin x, & 0 < x < \pi, \\ 0, & \text{otherwise,} \end{cases}$ (f) $\begin{cases} -\dfrac{1}{x}, & |x| \ge 1, \\ \dfrac{2}{1+x^2}, & |x| < 1. \end{cases}$

3.2.15. Graph the following piecewise continuous functions. List all discontinuities and jump magnitudes.

(a) $\begin{cases} e^x, & 1 < |x| < 2, \\ 0, & \text{otherwise,} \end{cases}$ (b) $\begin{cases} \sin x, & 0 < x < \frac{1}{2}\pi, \\ 0, & \text{otherwise,} \end{cases}$ (c) $\begin{cases} \dfrac{\sin x}{x}, & 0 < |x| < 2\pi, \\ 1, & x = 0, \\ 0, & \text{otherwise,} \end{cases}$

(d) $\begin{cases} x & |x| \le 1, \\ x^2, & |x| > 1, \end{cases}$ (e) $\begin{cases} x, & -1 < x < 0, \\ \sin x, & 0 < x < \pi, \\ 0, & \text{otherwise,} \end{cases}$ (f) $\begin{cases} -\dfrac{1}{x}, & |x| \ge 1, \\ \dfrac{2}{1+x^2}, & |x| < 1. \end{cases}$

3.2.16. Are the functions in Exercises 3.2.14 and 3.2.15 piecewise $C^1$? If so, list all corners.

3.2.17. Prove that the $n^{\text{th}}$ order *ramp function* $\rho_n(x-\xi) = \begin{cases} \dfrac{(x-\xi)^n}{n!}, & x > \xi, \\ 0, & x < \xi, \end{cases}$ is piecewise $C^k$ for any $k \ge 0$.

3.2.18. Is $x^{1/3}$ piecewise continuous? piecewise $C^1$? piecewise $C^2$?

3.2.19. Answer Exercise 3.2.18 for

(a) $\sqrt{|x|}$, (b) $\dfrac{1}{x}$, (c) $e^{-1/|x|}$, (d) $x^3 \sin \dfrac{1}{x}$, (e) $|x|^3$, (f) $|x|^{3/2}$.

3.2.20. (a) Give an example of a function that is continuous but not piecewise $C^1$.
(b) Give an example that is piecewise $C^1$ but not piecewise $C^2$.

3.2.21. (a) Prove that the sum $f + g$ of two piecewise continuous functions is piecewise continuous. (b) Where are the jump discontinuities of $f + g$? What are the jump magnitudes? (c) Check your result by summing the functions in parts (a) and (b) of Exercise 3.2.14.

3.2.22. Give an example of two piecewise continuous (but not continuous) functions $f, g$ whose sum $f + g$ is continuous. Can you characterize all such pairs of functions?

3.2.23. (a) Prove that if $f(x)$ is piecewise continuous on $[-\pi, \pi]$, then its $2\pi$–periodic extension is piecewise continuous on all of $\mathbb{R}$. Where are its jump discontinuities and what are their magnitudes? (b) Similarly, prove that if $f(x)$ is piecewise $C^1$, then its periodic extension is piecewise $C^1$. Where are the corners?

3.2.24. *True or false*: (a) If $f(x)$ is a piecewise continuous function, its absolute value $|f(x)|$ is piecewise continuous. If true, what are the jumps and their magnitudes?
(b) If $f(x)$ is piecewise $C^1$, then $|f(x)|$ is piecewise $C^1$. If true, what are the corners?

## The Convergence Theorem

3.2.25. (a) Sketch the $2\pi$–periodic half-wave $f(x) = \begin{cases} \sin x, & 0 < x \leq \pi, \\ 0, & -\pi \leq x < 0. \end{cases}$ (b) Find its Fourier series. (c) Graph the first five Fourier sums and compare with the function. (d) Discuss convergence of the Fourier series.

3.2.26. Answer Exercise 3.2.25 for the cosine half-wave $f(x) = \begin{cases} \cos x, & 0 < x \leq \pi, \\ 0, & -\pi \leq x < 0. \end{cases}$

3.2.27. (a) Find the Fourier series for $f(x) = e^x$. (b) For which values of $x$ does the Fourier series converge? Is the convergence uniform? (c) Graph the function it converges to.

3.2.28. (a) Use a graphing package to investigate the Gibbs phenomenon for the Fourier series (3.37) of the function $x$. Determine the amount of overshoot of the partial sums at the discontinuities. (b) How many terms do you need to approximate the function to within two decimal places at $x = 2.0$? At $x = 3.0$?

3.2.29. Use the Fourier series (3.49) for the step function to rederive Gregory's series (3.43).

3.2.30. Suppose $a_k, b_k$ are the Fourier coefficients of the function $f(x)$. (a) To which function does the Fourier series $\dfrac{a_0}{2} + \sum_{k=1}^{\infty} \left[\, a_k \cos 2kx + b_k \sin 2kx \,\right]$ converge? *Hint*: The answer is *not* $f(2x)$. (b) Test your answer with the Fourier series (3.37) for $f(x) = x$.

## Even and odd functions

3.2.31. Are the following functions even, odd, or neither?
(a) $x^2$, (b) $e^x$, (c) $\sinh x$, (d) $\sin \pi x$, (e) $\dfrac{1}{x}$, (f) $\dfrac{1}{1+x^2}$, (g) $\tan^{-1} x$.

3.2.32. Prove that (a) the sum of two even functions is even; (b) the sum of two odd functions is odd; (c) *every* function is the sum of an even and an odd function.

3.2.33. Prove (a) Lemma 3.12; (b) Lemma 3.13.

3.2.34. If $f(x)$ is odd, is $f'(x)$ (*i*) even? (*ii*) odd? (*iii*) neither? (*iv*) could be either?

3.2.35. If $f'(x)$ is even, is $f(x)$ (*i*) even? (*ii*) odd? (*iii*) neither? (*iv*) could be either? How do you reconcile your answer with Exercise 3.2.34?

3.2.36. Answer Exercise 3.2.34 for $f''(x)$.

3.2.37. *True or false*: (a) If $f(x)$ is odd, its $2\pi$–periodic extension is odd. (b) If the $2\pi$–periodic extension of $f(x)$ is odd, then $f(x)$ is odd.

3.2.38. Let $\widetilde{f}(x)$ denote the odd, $2\pi$–periodic Fourier extension of a function $f(x)$ defined on $[0, \pi]$. Explain why $\widetilde{f}(k\pi) = 0$ for any integer $k$.

3.2.39. Construct and graph the even and odd $2\pi$–periodic extensions of the function $f(x) = 1 - x$. What are their Fourier series? Discuss convergence of each.

3.2.40. Find the Fourier series and discuss convergence for: (a) the box function $b(x) = \begin{cases} 1, & |x| < \frac{1}{2}\pi, \\ 0, & \frac{1}{2}\pi < |x| < \pi, \end{cases}$ (b) the hat function $h(x) = \begin{cases} 1 - |x|, & |x| < 1, \\ 0, & 1 < |x| < \pi. \end{cases}$

3.2.41. Find the Fourier sine and cosine series of the following functions. Then graph the function to which the series converges. (a) 1, (b) $\cos x$, (c) $\sin^3 x$, (d) $x(\pi - x)$.

3.2.42. Find the Fourier series of the hyperbolic functions $\cosh mx$ and $\sinh mx$.

3.2.43. (a) Find the Fourier cosine series of the function $|\sin x|$.

(b) Use the series to evaluate the sums $\sum_{k=1}^{\infty}(4k^2-1)^{-1}$ and $\sum_{k=1}^{\infty}(-1)^{k-1}(4k^2-1)^{-1}$.

3.2.44. *True or false*: The sum of the Fourier cosine series and the Fourier sine series of the function $f(x)$ is the Fourier series for $f(x)$. If false, what function is represented by the combined Fourier series?

3.2.45. (a) Show that if a function is periodic of period $\pi$, then its Fourier series contains only even terms, i.e., $a_k = b_k = 0$ whenever $k = 2j+1$ is odd. (b) What if the period is $\frac{1}{2}\pi$?

3.2.46. Under what conditions on $f(x)$ does its Fourier sine series contain only even terms, i.e., its Fourier sine coefficients $b_k = 0$ whenever $k$ is odd?

3.2.47. Graph the partial sums $s_3(x), s_5(x), s_{10}(x)$ of the Fourier series (3.55). Do you notice a Gibbs phenomenon? If so, what is the amount of overshoot?

3.2.48. Explain why, in the case of the step function $\sigma(x)$, all its Fourier cosine coefficients vanish, $a_k = 0$, except for $a_0 = 1$.

3.2.49. How many terms do you need to sum in (3.56) to correctly approximate $\pi$ to two decimal digits? To ten digits?

3.2.50. Prove that $\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \frac{1}{49} - \cdots = \frac{\pi^2}{12}.$

## Complex Forier Series

3.2.51. Find the complex Fourier series of the following functions: (a) $\sin x$, (b) $\sin^3 x$, (c) $x$, (d) $|x|$, (e) $|\sin x|$, (f) $\operatorname{sign} x$, (g) the ramp function $\rho(x) = \begin{cases} x, & x \geq 0, \\ 0, & x \leq 0. \end{cases}$

3.2.52. Let $-\pi < \xi < \pi$. Determine the complex Fourier series for the shifted step function $\sigma(x-\xi)$, and graph the function it converges to.

3.2.53. Let $a \in \mathbb{R}$. Find the real form of the Fourier series for the exponential function $e^{ax}$:
(a) by breaking up the complex series (3.68) into its real and imaginary parts;
(b) by direct evaluation of the real coefficients via their integral formulae (3.35).
Make sure that your results agree!

3.2.54. Prove that $\coth \pi = \dfrac{1}{\pi} + \dfrac{2}{\pi}\left(\dfrac{1}{1+1^2} + \dfrac{1}{1+2^2} + \dfrac{1}{1+3^2} + \cdots\right)$, where $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$ is the hyperbolic cotangent function.

3.2.55. (a) Find the complex Fourier series for $x\, e^{\,\mathrm{i}\, x}$.
(b) Use your result to write down the real Fourier series for $x \cos x$ and $x \sin x$.

3.2.56. Prove that if $f(x) = \sum_{k=m}^{n} r_k\, e^{\,\mathrm{i}\, k x}$ is a complex trigonometric polynomial, with $-\infty < m \le n < \infty$, then its Fourier coefficients are $c_k = \begin{cases} r_k, & m \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$

3.2.57. *True or false*: If the complex function $f(x) = g(x) + \mathrm{i}\, h(x)$ has Fourier coefficients $c_k$, then $g(x) = \operatorname{Re} f(x)$ and $h(x) = \operatorname{Im} f(x)$ have, respectively, complex Fourier coefficients $\operatorname{Re} c_k$ and $\operatorname{Im} c_k$.

3.2.58. Let $f(x)$ be $2\pi$–periodic. Explain how to construct the complex Fourier series for $f(x-a)$ from that of $f(x)$.

3.2.59. (a) Show that if $c_k$ are the complex Fourier coefficients for $f(x)$, then the Fourier coefficients of $\widetilde{f}(x) = f(x)\, e^{\,\mathrm{i}\, x}$ are $\widetilde{c}_k = c_{k-1}$. (b) Let $m$ be an integer. Which function has complex Fourier coefficients $\widehat{c}_k = c_{k+m}$? (c) If $a_k, b_k$ are the Fourier coefficients of the real function $f(x)$, what are the Fourier coefficients of $f(x) \cos x$ and $f(x) \sin x$?

3.2.60. Can you recognize whether a function is real by looking at its complex Fourier coefficients?

3.2.61. Can you characterize the complex Fourier coefficients of an even function? an odd function?

3.2.62. What does it mean for a doubly infinite series $\sum_{k=-\infty}^{\infty} c_k$ to converge? Be precise!

## 3.3 Differentiation and Integration

3.3.1. Starting with the Fourier series (3.49) for the step function $\sigma(x)$, use integration to:
(a) Find the Fourier series for the ramp function $\rho(x) = \begin{cases} x, & x > 0, \\ 0, & x < 0. \end{cases}$
(b) Then, find the Fourier series for the second-order ramp function $\rho_2(x) = \begin{cases} \frac{1}{2} x^2, & x > 0, \\ 0, & x < 0. \end{cases}$

3.3.2. Find the Fourier series for the function $f(x) = x^3$. If you differentiate your series, do you recover the Fourier series for $f'(x) = 3x^2$? If not, explain why not.

3.3.3. Answer Exercise 3.3.2 when $f(x) = x^4$.

3.3.4. Use Theorem 3.20 to construct the Fourier series for (a) $x^3$, (b) $x^4$.

3.3.5. Write down the identities obtained by substituting $x = 0, \frac{1}{2}\pi$, and $\frac{1}{3}\pi$ in the Fourier series (3.74).

3.3.6. Suppose $f(x)$ is a $2\pi$–periodic function with complex Fourier coefficients $c_k$, and $g(x)$ is a $2\pi$–periodic function with complex Fourier coefficients $d_k$. (a) Find the Fourier coefficients $e_k$ of their *periodic convolution* $f(x) * g(x) = \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy$.
(b) Find the complex Fourier series for the periodic convolution of $\cos 3x$ and $\sin 2x$.
(c) Answer part (b) for the functions $x$ and $\sin 2x$.

3.3.7. Suppose $f$ is piecewise continuous on $[-\pi, \pi]$. Prove that the mean of the integrated function $g(x) = \int_0^x f(y)\, dy$ equals $\frac{1}{2}\int_{-\pi}^{\pi} \left( \operatorname{sign} x - \frac{x}{\pi} \right) f(x)\, dx$.

3.3.8. Suppose the $2\pi$–periodic extension of $f(x)$ is continuous and piecewise $C^1$. Prove directly from the formulas (3.35) that the Fourier coefficients of its derivative $\widetilde{f}(x) = f'(x)$ are, respectively, $\widetilde{a}_k = k\, b_k$ and $\widetilde{b}_k = -k\, a_k$, where $a_k, b_k$ are the Fourier coefficients of $f(x)$.

3.3.9. Explain how to integrate a complex Fourier series (3.64). Under what conditions is your formula valid?

3.3.10. The initial value problem $\frac{d^2u}{dt^2} + u = f(t)$, $u(0) = 0$, $\frac{du}{dt}(0) = 0$, describes the forced motion of an initially motionless unit mass attached to a unit spring.
(a) Solve the initial value problem when $f(t) = \cos kt$ and $f(t) = \sin kt$ for $k = 0, 1, \ldots$.
(b) Assuming that the forcing function $f(t)$ is $2\pi$–periodic, write out its Fourier series, and then use your result from part (b) to write out a series for the solution $u(t)$.
(c) Under what conditions is the result a convergent Fourier series, and hence the solution $u(t)$ remains $2\pi$–periodic?
(d) Explain why $f(t)$ induces a resonance of the mass-spring system if and only if its Fourier coefficients of order 1 are not both zero: $a_1^2 + b_1^2 \neq 0$.

## 3.4 Change of Scale

3.4.1. Let $f(x) = x^2$ for $0 \le x \le 1$. Find its (a) Fourier sine series; (b) Fourier cosine series.

3.4.2. Find the Fourier sine series and the Fourier cosine series of the following functions defined on the interval $[0,1]$; then graph the function to which the series converges:
$(a)$ 1, $(b)$ $\sin \pi x$, $(c)$ $\sin^3 \pi x$, $(d)$ $x(1-x)$.

3.4.3. Find the Fourier series for the following functions on the indicated intervals, and graph the function that the Fourier series converges to.
$(a)$ $|x|$, $-3 \le x \le 3$, $(b)$ $x^2-4$, $-2 \le x \le 2$, $(c)$ $e^x$, $-10 \le x \le 10$,
$(d)$ $\sin x$, $-1 \le x \le 1$, $(e)$ $\sigma(x)$, $-2 \le x \le 2$.

3.4.4. For each of the functions in Exercise 3.4.3, write out the differentiated Fourier series, and determine whether it converges to the derivative of the original function.

3.4.5. Find the Fourier series for the integral of each of the functions in Exercise 3.4.3.

3.4.6. Write down formulas for the Fourier series of both even and odd functions on $[-\ell, \ell]$.

3.4.7. Let $f(x)$ be a continuous function on $[0, \ell]$.
$(a)$ Under what conditions is its odd $2\ell$–periodic extension also continuous?
$(b)$ Under what conditions is its odd extension also continuously differentiable?

3.4.8. $(a)$ Write down the formulae for the Fourier series for a function $f(x)$ defined on the interval $0 \le x \le 2\pi$. $(b)$ Use your formula in the case $f(x) = x$. Is the result the same as (3.37)? Explain, and, if different, discuss the connection between the two Fourier series.

3.4.9. Find the Fourier series for the function $f(x) = x$ on the interval $1 \le x \le 2$ using the two different methods described in the last paragraph of this subsection. Are your Fourier series the same? Explain. Graph the functions that the Fourier series converge to.

3.4.10. Answer Exercise 3.4.9 when $f(x) = \sin x$ on the interval $\pi \le x \le 2\pi$.

## 3.5 Convergence of Fourier Series

3.5.1. Consider the following sequence of planar vectors $\mathbf{v}^{(n)} = \left(1 - \frac{1}{n}, e^{-n}\right)$, $n = 1, 2, 3, \ldots$.
Prove that $\mathbf{v}^{(n)}$ converges to $v^\star = (1, 0)$ as $n \to \infty$ by showing that: $(a)$ the individual components converge; $(b)$ the Euclidean norms converge: $\| \mathbf{v}^{(n)} - v^\star \|_2 \to 0$.

3.5.2. Which of the following sequences of vectors converge as $n \to \infty$? What is the limit? (a) $\left( \dfrac{1}{1+n^2}, \dfrac{n^2}{1+2n^2} \right)$, (b) $(\cos n, \sin n)$, (c) $\left( \dfrac{\cos n}{n}, \dfrac{\sin n}{n} \right)$, (d) $\left( \cos \frac{1}{n}, \sin \frac{1}{n} \right)$, (e) $\left( \frac{1}{n} \cos \frac{1}{n}, \frac{1}{n} \sin \frac{1}{n} \right)$, (f) $\left( e^{-n}, n\, e^{-n}, n^2 e^{-n} \right)$, (g) $\left( \dfrac{\log n}{n}, \dfrac{(\log n)^2}{n^2}, \dfrac{(\log n)^3}{n^3} \right)$, (h) $\left( \dfrac{1-n}{1+n}, \dfrac{1-n}{1+n^2}, \dfrac{1-n^2}{1+n^2} \right)$, (i) $\left( \left(1+\frac{1}{n}\right)^n, \left(1-\frac{1}{n}\right)^{-n} \right)$, (j) $\left( \dfrac{e^n-1}{n}, \dfrac{\cos n - 1}{n^2} \right)$, (k) $\left( n\left(e^{1/n}-1\right), n^2\left(\cos \frac{1}{n} - 1\right) \right)$.

3.5.3. Which of the following sequences of functions converge pointwise for $x \in \mathbb{R}$ as $n \to \infty$? What is the limit? (a) $1 - \dfrac{x^2}{n^2}$, (b) $e^{-nx}$, (c) $e^{-nx^2}$, (d) $|x-n|$, (e) $\dfrac{1}{1+(x-n)^2}$, (f) $\begin{cases} 1, & x < n, \\ 2, & x > n, \end{cases}$ (g) $\begin{cases} n^2, & \frac{1}{n} < x < \frac{2}{n}, \\ 0, & \text{otherwise}, \end{cases}$ (h) $\begin{cases} x, & |x| < n, \\ n x^{-2}, & |x| \geq n. \end{cases}$

3.5.4. Prove that the sequence $v_n(x) = \begin{cases} 1, & 0 < x < \frac{1}{n}, \\ 0, & \text{otherwise}, \end{cases}$ converges pointwise, but not uniformly, to the zero function.

3.5.5. Which of the following sequences of functions converge pointwise to the zero function for all $x \in \mathbb{R}$? Which converge uniformly? (a) $-\dfrac{x^2}{n^2}$, (b) $e^{-n|x|}$, (c) $x\,e^{-n|x|}$, (d) $\dfrac{1}{n(1+x^2)}$, (e) $\dfrac{1}{1+(x-n)^2}$, (f) $|x-n|$, (g) $\begin{cases} \frac{1}{n}, & 0 < |x| < n, \\ 0, & \text{otherwise}, \end{cases}$ (h) $\begin{cases} n, & 0 < |x| < \frac{1}{n}, \\ 0, & \text{otherwise}, \end{cases}$ (i) $\begin{cases} x/n, & |x| < 1, \\ 1/(nx), & |x| \geq 1. \end{cases}$

3.5.6. Does the sequence $v_n(x) = n\,x\,e^{-n x^2}$ converge pointwise to the zero function for $x \in \mathbb{R}$? Does it converge uniformly?

3.5.7. Answer Exercise 3.5.6 when (a) $v_n(x) = x\,e^{-nx^2}$, (b) $v_n(x) = \begin{cases} 1, & n < x < n+1, \\ 0, & \text{otherwise}, \end{cases}$ (c) $v_n(x) = \begin{cases} 1, & n < x < n + 1/n, \\ 0, & \text{otherwise}, \end{cases}$ (d) $v_n(x) = \begin{cases} 1/n, & n < x < 2n, \\ 0, & \text{otherwise}, \end{cases}$ (e) $v_n(x) = \begin{cases} 1/\sqrt{n}, & n < x < 2n, \\ 0, & \text{otherwise}, \end{cases}$ (f) $v_n(x) = \begin{cases} n^2 x^2 - 1, & -1/n < x < 1/n, \\ 0, & \text{otherwise}. \end{cases}$

3.5.8. (a) What is the limit of the functions $v_n(x) = \tan^{-1} n x$ as $n \to \infty$? (b) Is the convergence uniform on all of $\mathbb{R}$? (c) on the interval $[-1,1]$? (d) on the subset $\{x \geq 1\}$?

3.5.9. *True or false*: If $p_n(x)$ is a sequence of polynomials that converge pointwise to a polynomial $p_\star(x)$, then the convergence is uniform.

3.5.10. Suppose $v_n(x)$ are continuous functions such that $v_n \to v_\star$ pointwise on all of $\mathbb{R}$. *True or false*: (*a*) $v_n - v_\star \to 0$ pointwise; (*b*) if $v_\star(x) \neq 0$ for all $x$, then $\dfrac{v_n}{v_\star} \to 1$ pointwise.

3.5.11. Which of the following series satisfy the $M$–test and hence converge uniformly on the interval $[0,1]$? (*a*) $\displaystyle\sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$, (*b*) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin kx}{k}$, (*c*) $\displaystyle\sum_{k=1}^{\infty} x^k$, (*d*) $\displaystyle\sum_{k=1}^{\infty} (x/2)^k$, (*e*) $\displaystyle\sum_{k=1}^{\infty} \frac{e^{kx}}{k^2}$, (*f*) $\displaystyle\sum_{k=1}^{\infty} \frac{e^{-kx}}{k^2}$, (*g*) $\displaystyle\sum_{k=1}^{\infty} \frac{e^{x/k}-1}{k}$.

3.5.12. Prove that the power series $\displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k(k+1)}$ converges uniformly for $-1 \le x \le 1$.

3.5.13. (*a*) Prove the following result: Suppose $|\,g(x)\,| \le M$ for all $x \in I$. If (3.93) is a uniformly convergent series on $I$, so is the term-wise product (3.94).
(*b*) Find a counterexample when $g(x)$ is not uniformly bounded.

3.5.14. Suppose each $u_k(x)$ is continuous, and the series $\displaystyle\sum_{k=1}^{\infty} u_k(x) = f(x)$ converges uniformly on the bounded interval $a \le x \le b$. Prove that the integrated series (3.95) is uniformly convergent.

3.5.15. Prove that if $\displaystyle\sum_{k=1}^{\infty} \sqrt{a_k^2 + b_k^2} < \infty$, then the real Fourier series (3.34) converges uniformly to a continuous $2\pi$–periodic function.

3.5.16. Suppose $\displaystyle\sum_{k=1}^{\infty} |\,a_k\,| < \infty$ and $\displaystyle\sum_{k=1}^{\infty} |\,b_k\,| < \infty$. Does the conclusion of Exercise 3.5.15 still hold?

3.5.17. Explain why you only need check the inequalities (3.91) for all sufficiently large $k \gg 0$ in order to use the Weierstrass $M$–test.

3.5.18. Suppose we say that a sequence of vectors $\mathbf{v}^{(k)} \in \mathbb{R}^m$ converges *uniformly* to $v^\star \in \mathbb{R}^m$ if, for every $\varepsilon > 0$, there is an $N$, depending only on $\varepsilon$, such that $|\,v_i^{(k)} - v_i^\star\,| < \varepsilon$, for all $k \ge N$ and all $i = 1, \ldots, m$. Prove that *every* convergent sequence of vectors converges uniformly.

3.5.19. (a) Let $S = \{x_1, x_2, x_3, \ldots\} \subset \mathbb{R}$ be a countable set. Prove that $S$ has measure zero by showing that, for every $\varepsilon > 0$, there exists a collection of open intervals $I_1, I_2, I_3, \ldots \subset \mathbb{R}$, with respective lengths $\ell_1, \ell_2, \ell_3, \ldots$, such that $S \subset \bigcup I_j$, while the total length $\sum \ell_j = \varepsilon$.
(b) Explain why the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ is dense but nevertheless has measure zero.

## Smooth and Decay

3.5.20. (a) Prove that the complex Fourier series $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}\, e^{\,\mathrm{i}\,k\,x}$ converges uniformly on the interval $[-\pi, \pi]$. (b) Is the sum $f(x)$ continuous? Why or why not?
(c) Is $f(x)$ continuously differentiable? Why or why not?

3.5.21. First, without explicitly evaluating them, how fast do you expect the Fourier coefficients of the following functions to go to zero as $k \to \infty$? Then prove your claim by evaluating the coefficients.
(a) $x - \pi$, (b) $|x|$, (c) $x^2$, (d) $x^4 - 2\pi^2 x^2$, (e) $\sin^2 x$, (f) $|\sin x|$.

3.5.22. Using the criteria of Theorem 3.31, determine how many continuous derivatives the functions represented by the following Fourier series have:
(a) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{e^{\,\mathrm{i}\,k\,x}}{1+k^4}$, (b) $\displaystyle\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{\,\mathrm{i}\,k\,x}}{k^2+k^5}$, (c) $\displaystyle\sum_{k=-\infty}^{\infty} e^{\,\mathrm{i}\,k\,x-k^2}$, (d) $\displaystyle\sum_{k=0}^{\infty} \frac{e^{\,\mathrm{i}\,k\,x}}{k+1}$,
(e) $\displaystyle\sum_{k=-\infty}^{\infty} \frac{e^{\,\mathrm{i}\,k\,x}}{|\,k\,|!}$, (f) $\displaystyle\sum_{k=1}^{\infty} \left(1-\cos\frac{1}{k^2}\right) e^{\,\mathrm{i}\,k\,x}$.

3.5.23. Discuss convergence of each of the following Fourier series. How smooth is the sum? Graph the partial sums to obtain a reasonable approximation to the graph of the summed series. How many summands are needed to obtain accuracy in the second decimal digit over the entire interval? Point out discontinuities, corners, and other features that you observe.
(a) $\displaystyle\sum_{k=0}^{\infty} e^{-k} \cos k\,x$, (b) $\displaystyle\sum_{k=0}^{\infty} \frac{\cos k\,x}{k+1}$, (c) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k\,x}{k^{3/2}}$, (d) $\displaystyle\sum_{k=1}^{\infty} \frac{\sin k\,x}{k^3+k}$.

3.5.24. Prove that if $|\,a_k\,|,\ |\,b_k\,| \le M\,k^{-\alpha}$ for some $M > 0$ and $\alpha > n+1$, then the real Fourier series (3.34) converges uniformly to an $n$–times continuously differentiable $2\pi$–periodic function $f \in \mathrm{C}^n$.

3.5.25. Give a simple explanation of why, if the Fourier coefficients $a_k = b_k = 0$ for all sufficiently large $k \gg 0$, then the Fourier series converges to an analytic function.

## Hilbert Space

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